THE NUMBER e

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1. The Greek beginning. The distinguished American mathematician, Benjamin Peirce, was wont to find all of analysis in the equation

$$i^{-i} = \sqrt{e^{\pi}}.$$

In fact, he had his picture taken in front of a blackboard on which this mystic formula, in somewhat different shape, was inscribed. He would say to his hearers, "Gentlemen, we have not the slightest idea of what this equation means, but we may be sure that it means something very important."

With regard to the symbols which appear in this charm, there is a vast literature connected with π; and i, when written $\sqrt{-1}$, leads into the broad field of analysis in the complex domain; but it seems surprisingly difficult to find a connected account of e.

I think we may make a fair beginning with the twelfth proposition of the Second Book of Apollonius, *Conics*, which tells us that if from a point on a hyperbola lines be drawn in given directions to meet the asymptotes, the product of the two distances is independent of the position of the point chosen on the curve. This theorem is more general than we shall need to arrive at the number e, and it is not original with Apollonius. Let us confine ourselves to the very special case where the hyperbola is rectangular, and we draw to each asymptote a line parallel to the other. When x and y are distances, we may write

$$xy = 1. \tag{1}$$

It is intriguing to inquire who first discovered the theorem which leads to this equation. In the commentary of Eutocius on the Sphere and Cylinder of Archimedes [1], we come to a discussion of the classical problem of inserting two mean proportionals between two given lengths. In one solution, which he labels "ut Menaechmus," we have what amounts to the equations

$$\begin{gathered} a/x = x/y = y/b; \\ y^2 = bx; \qquad xy = ab. \end{gathered} \tag{2}$$

He goes on to seek the intersection of a parabola and a hyperbola.

Eutocius' statement would place the theorem very early in the history of the conics, for Menaechmus is usually regarded as the discoverer or inventor of these curves, although this ascription is by no means certain. Allman writes [2], "It is much to be regretted that the two solutions of Menaechmus have not been transmitted to us in their original form. That they have been altered either by Eutocius or by some author whom he followed appears not only in the employment in these solutions of the terms parabola and hyperbola, as has frequently been pointed out, but more from the fact that the language used in

them is, in character, altogether that of Apollonius." A similar doubt is shown in Loria [3]. On the other hand, Heath is perfectly definite on this point; he states, "This property in the particular case of the rectangular hyperbola was known to Menaechmus" [4].

But there is another reason for doubting the ascription to Menaechmus, aside from the linguistic objection. The classical Greek discussion of the conics always corresponds to our analysis when the axes are a tangent and the diameter through the point of contact, and with these data proofs are not simple. Heath, following Zeuthen, shows the fact that the hyperbola can be written immediately in the form (1) if we start with a technique like ours, that is, when the axes are a pair of conjugate diameters [5]. That is perfectly true, but the Greeks made surprisingly little study of the conics when expressed in this form more familiar to us; Apollonius comes to it quite late. It seems to me altogether doubtful that the first discoverer of the curves should have been able to make the transition.

2. Grégoire de St. Vincent. If we grant that the Greek mathematicians, perhaps Menaechmus, were familiar with the fundamental property of the rectangular hyperbola expressed in (1), what has this to do with *e*? We must look ahead some two thousand years to that original writer whose name appears at the head of this paragraph. In 1647, he published his fundamental *Prologomena a Santo Vincento, Opus geometricum quadraturae circuli et sectionum coni.* This I have not seen in its original form, but the content is given at great length by Bopp in [6]. Here is the general scheme. We take the hyperbola

$$xy = 1. \tag{1}$$

On the x axis we take n equivalent rectangles whose bases are

$$P_0P_1,\ P_1P_2,\ \cdots,\ P_{n-1}P_n,$$

while each has an upper vertex on the curve Q_i. Then,

$$P_0P_1 \cdot P_0Q_0 = P_1P_2 \cdot P_1Q_1 = P_2P_3 \cdot P_2Q_2 = \cdots,$$

and

$$\frac{P_0Q_0}{P_1Q_1} = \frac{P_1P_2}{P_0P_1}; \qquad \frac{P_1Q_1}{P_2Q_2} = \frac{P_2P_3}{P_1P_2} \tag{2}$$

but

$$OP_0 \cdot P_0Q_0 = OP_1 \cdot P_1Q_1 = OP_2 \cdot P_2Q_2 \cdots,$$

so that

$$\frac{OP_0}{OP_1} = \frac{P_1Q_1}{P_0Q_0} = \frac{P_0P_1}{P_1P_2} = \frac{OP_1}{OP_2}, \quad \text{by composition.}$$

If

$$OP_1 = \rho OP_0 \qquad \text{then} \qquad OP_j = \rho^j OP_0. \tag{3}$$

St. Vincent even treats the case where OP_0 and OP_j are incommensurable, but we need not follow him here.

The importance of this equation was early recognized, because of its connection with logarithms which were based on the relation of arithmetical and geometrical series. There is a good deal to be said in favor of the thesis that the credit for relating the rectangular hyperbola with logarithms is due to Sarasa. I have not seen his work, but like Cantor, I rely on Kästner. In 1649, Sarasa published *Solutio Problematis a R. P. Marino Mersenno propositi.* This was concerned with the problem: Given three positive quantities and the logarithms of two of them, find the logarithm of the third. Kästner writes [7], "Zu ihrer Beanwortung brang Sarasa drey Saetze aus des Gregorius Buche von der Hyperbel bey, die betreffen Flaechen der Hyperbel an der Aysmptoten, Sarasa erinnert wie das mit Logarithmen zusammenhangt." Cantor's view is similar [8]; he states, "Mit andern Worten, Gregorius hatte das Auftreten von Logarithmen bei der erhähnten Flächenraumen erkannt, wen auch nicht mit Namen genannt. Letzteres that Sarasa, und darin liegt das wirkliche Verdienst seiner Stratschrift."

A contrary view is expressed by Charles Hutton [9] in the words, "As to the first remarks on the analogy between logarithms and hyperbolic spaces, it having been shown by Gregory St. Vincent . . . that if an asymptote be divided into parts in geometrical progression, and from the points of division ordinates be drawn parallel to the other asymptote, they will divide the space between the asymptote and the curve into equal portions, from hence it was shown by Mersenne, that by taking continual sums of these parts there would be obtained areas in arithmetical progression which therefore were analogous to a system of logarithms."

This may be true, but I must point out that whereas St. Vincent published the work referred to above in 1647, Mersenne died in the middle of 1648, and the dates of all of his mathematical writings which I have seen were much earlier. However, St. Vincent's work was certainly well observed. We find Wallis writing in 1658 to Lord Brouncker [10], "Sumptis (in Asymptoto) rectis *NH, NI, NK, NQ, NL, NM* geometrice proportionalibus, in punctis *H, I, K, Q, L, M,* ducantur rectae parallelae alteri Asymptoto, spatium Hyperbolicum *A B H M* in quinque partes dividi ostendit Gregor de Sancto Vincento (si memini) decimo."

3. The introduction of logarithms. The actual word logarithm occurs again in an account of Gregory's *Vera circuli et hyperbolae quadratura,* which was published in Padua in 1667 and laid before the Royal Society [11]. Here we read, "And lastly by the same method he calculates both the logarithm of any natural number, and, vice versa, the natural number of any given logarithm." Perhaps

the wisest word on the subject has been pronounced by the kindly old writer Montucla [12], "Au reste la découverte de cette propriété est revindiquée par divers autres géomètres." Among these I surely must mention Christian Huygens, who acknowledges the work of St. Vincent, even though he does not claim for himself the discovery of the relation between the hyperbola and logarithms. This is admirably set forth in [13], first in a French account, then Huygens' own Latin. He finds the areas bounded by the x axis, which is an asymptote, the curve and ordinates. Two such areas terminating by the same ordinate of 1 are

$$\frac{\text{Area } FGDE}{\text{Area } ABDE} = \frac{\log_e FG}{\log_e 10} = \log_{10} FG.$$

Huygens divides numerator and denominator by 32, which amounts to finding the 32nd root of each area, but this has the effect of so far closing up the figure that we may safely replace the hyperbola by a parabola whose outside area is known. He checks by finding a very good value for $\log_{10} 2$.

In the same year, 1661, Huygens finds another curve which he calls logarithmic but we should probably call it exponential. This curve has the property that the ordinate corresponding to the point mid-way between two given points of the x-axis is the mean proportional between their ordinates. The equation of the curve is $y = ka^x$. Huygens takes

$$y = 2^{x/x_0}; \qquad x = \frac{\log y}{\log 2} x_0. \tag{4}$$

The constant subtangent is

$$\frac{ydx}{dy} = \frac{x_0}{\log_e 2}. \tag{5}$$

Huygens takes

$$x_0 = 10^n \log_{10} 2.$$

This gives for the constant subtangent

$$\log_{10} e = .43429448190325180,$$

"qualium logarithmus binarij est"

$$.30102995663981195.$$

These numbers had long been known as they had appeared, for instance, in Briggs' *Arithmetica logarithmica* of 1624, pages 10 and 14. As a matter of fact, there appeared in 1618 a second edition of Wright's translation of Napier's *Mirifici Logarithmorum Canonis Descriptio* which contained an appendix, probably written by Oughtred, giving the natural logarithms of various numbers

from 100,000 to 900,000. This is probably the earliest table of natural logarithms, although a very similar table by John Spidell appeared in 1619 [14].

The astonishing thing about all of those writers who connected logarithms with hyperbolic areas is their lack of interest in what we should call the base. Napier began by considering the relation between an arithmetical and a geometric series. A geometric series consists in successive powers of one number. What is that number? Or given a set of logarithms, what number has the logarithm 1? I mentioned that Briggs gave the logarithm of *e*, to the base 10 but I find no mention of *e* itself. Of course, we might write

$$10^n \log_{10} \frac{10^n + \Delta x}{10^n} = 10^n \log_e \left(1 + \frac{\Delta x}{10^n}\right) \log_{10} e = \Delta x \log_{10} e + \cdots,$$

but *e* itself does not appear. The fact is that there was no comprehension that a logarithm was essentially an exponent. Tropfke is very explicit in this point; he writes, "Freilich dürfen wir nicht an die moderne Erklärung der Logarithmen denken, die in ihnen Potenzexponenten einer bestimmten Grundzahl erkennt. Diese Auffassung machte sich erst um die Mitte des achtenten Jahrhunderts geltend" [15]. This is perhaps too strong a statement, for in a note on the same page he quotes James Gregory (whom he calls David Gregory) as saying in his *Exercitationes Geometricae* of 1684, p. 14, "Exponentes sunt ut logarithmi." I have not been able to verify this, but we find in [16], "Si seriei Termonorum in Progressione geometrica ab 1 continue proportionalium, puta

1, 2, 4, 8, 16, 32, 64, etc.

accomedetur series Indicum, sive Exponentium, in progressione ab o continue procedentium, puta

0, 1, 2, 3, 4, 5, 6, etc.

Hos exponentes appelabant Logarithmos." We could not well ask for anything clearer or more explicit.

If most writers did not look on logarithms as exponents, how did they consider them? I think we find the clue in St. Vincent's identification of logarithms with hyperbolic areas, remembering that these were the days of Cavalieiri and Roberval, when an area was looked upon as the same thing as an infinite number of line segments, a very helpful if dangerous definition. We find Halley writing [17], "They may more properly be said to be numeri rationum exponentes, wherein we consider ratio as a quantity sui generis, beginning from the ratio of equality, or 1 to $1=0$, $\cdots$ and the rationes we suppose to be measured by the number of ratiunculae in each. Now these ratiunculae are in a continued scale of proportionals, infinite in number, between the two terms of the ratio, which infinite number of mean proportionals is to that infinite number of the like and equal ratiunculae between any other two terms as the logarithm of one ratio is to the logarithm of the other. Thus if we suppose there to be between 1 and 10

an infinite scale of mean proportionals whose number is 100000 ad infinitum, between 1 and 2 there shall be 30102 of said proportionals and between 1 and 3, 47712 of them which numbers therefore are the logarithms of the ratio of 1 to 10 1 to 2, and 1 to 3, and so properly called the logarithms of 10, 2, and 3."

It is hard to see how there could be a much worse explanation of logarithms for those who "make constant use of logarithms without having an adequate notion of them." The one certain thing seems to be that a logarithm is an infinite number. I suppose we might translate this into the form

$$\frac{b}{a} = \frac{a+r_1}{a} \cdot \frac{a+r_2}{a+r_1} \cdot \frac{a+r_3}{a+r_2}, \cdots, \frac{a+r_n}{a+r_{n-1}} \cdot \frac{b}{a+r_n}.$$

If

$$\frac{a+r_j}{a+r_{j-1}} = r, \qquad \frac{b}{a} = r^n.$$

Then n would be the logarithm.

4. Mercator, Newton, Leibniz. It is fair to say that such a definition of a logarithm was not original with Halley. We find Mercator writing in 1668, [18] "Est enim Logarithmus nihil aliud, quam numerus ratiuncularum contentarum in ratione quam absolutus quisque ad unitatem obtinet." I may mention also that this seems the first place where the words "logarithmus naturalis" are used. But the real significance of the article comes from the fact that instead of studying $\log x$ he takes up $\log (1+x)$, which enables him to start from 0. The article is not clearly written, so I follow the much clearer exposition in Wallis [19], which was published in the same year.

We study the area under the curve (1) from $x=1$ to $x=1+X$. We divide the length on the x-axis into n equal parts, each of length Δx. The abscissas are $1, 1+\Delta x, 1+2\Delta x, \cdots, 1+X$ and the corresponding ordinates are

$$1, \frac{1}{1+\Delta x}, \frac{1}{1+2\Delta x}, \cdots, \frac{1}{1+(n-1)\Delta x}.$$

The infinitesimal, rectangular areas are

$$\Delta x, \Delta x[1 - \Delta x + \Delta x^2 - \Delta x^3 + \cdots],$$
$$\Delta x[1 - (2\Delta x) + (2\Delta x)^2 - (2\Delta x)^3 + \cdots], \cdots.$$

Such infinite expansions were common in Wallis' work. The sum of these rectangular areas may be written

$$\Delta x[1 + 1 + 1 \cdots] - \Delta x[\Delta x + 2\Delta x + 3\Delta x + \cdots]$$
$$+ \Delta x[(\Delta x)^2 + (2\Delta x)^2 + (3\Delta x)^2 + \cdots] - \cdots.$$

Now $n\Delta x = X$, so we have

$$X - \Delta x^2[1 + 2 + 3 \cdots] + \Delta x^3[1^2 + 2^2 + 3^2 \cdots] - \cdots. \tag{6}$$

With regard to these sums, Wallace says [19], page 222, "quod ostendit ille prop XVI etque a me alibi demonstratum." A reference he makes to Mercator is not conclusive as the statement is sketchy; as to his own work I will follow [20], as I shall need that again. Here he is seeking the area under the curve $y=x^m$ from $x=0$ to $x=X$. His method is not perfectly clear, as he seems merely to generalize by analogy from cases worked out earlier, but what he does is essentially the following:

We take N equal lengths from 0 to $N\Delta x=X$. We have a set of rectangles whose combined areas are

$$\Delta x[0^m + (\Delta x)^m + (2\Delta x)^m + (3\Delta x)^m + \cdots].$$

Let us assume that $0^m+1^m+\cdots+(N-1)^m=\alpha N^{m+1}+\beta N^m+\gamma N^{m-1}+\cdots$. Replacing N by $N+1$, and subtracting, we obtain

$$N^m = (m+1)\alpha N^m + bN^{m-1} + cN^{m-2}\cdots,$$

so

$$\alpha = \frac{1}{m+1}.$$

Substituting, and remembering that $N\Delta x=X$, there results

$$\text{Area} = \frac{X^{m+1}}{m+1} + \beta\Delta x X^m + \gamma\Delta x^2 X^{m-1}.$$

The limit of this as $N\to\infty$ is $X^{m+1}/m+1$, since $\Delta x\to 0$. We thus can substitute this result in (6), when $m=1, 2, 3, \cdots$, to obtain Mercator's famous formula:

$$\log(1+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \cdots. \tag{7}$$

A good deal has been written about this series, as we see from Mazeres and elsewhere. The obvious way to obtain the equation is to apply the calculus, so we now turn to see how this instrument was brought to bear. In 1669, a year after Mercator had published his work on logarithms [18], Newton sent to Collins his article, *De Analysi per aequationes numero terminorum Infinitas* [21]. This represents his first studies of areas under curves, which he had been working at for a year or two, but had not published. In fact, publication did not occur for a goodly number of years to come; there is, however, no question of giving his results precedence over those of Mercator. It begins as shown below:

Curvarum simplicium Quadratura

Reg. 1 Si $ax^{m/n} = y$, erit $\dfrac{an}{m+n} x^{(m+n)/n} =$ Area ABD.

I must speak further of this. In [22] we read on p. 176, "Dr. Wallis published in his Arithmetica infinitorum in the year 1655 and in the 59th Proposition of that Book, if the Abscissa of any curvilinear figure be called x and m and n be two Numbers, the ordinates erected at right Angles be $x^{n/m}$ the area of the Figure shall be $(n/m+n)x^{m+n/n}$. And this is assumed by Mr. Newton, upon which he founds his Quadrature of Curves. Dr. Wallis demonstrated this by steps in many particular Propositions and then connected all the Propositions into one by a Table of Cases. Mr. Newton reduced all Cases to One, with an indefinite Index, and at the end of his Compendium demonstrated it at once by his method of moments, he being the first who introduced indefinite Indices of Dignites into the Operations of Analysis." This is Newton's own statement of the case and must be taken as final. It is true that Wallis worked out a number of special cases in a manner not exactly like the method followed here, and did not use a literal exponent. The greater generality of Newton's formula is found by replacing x by $x^{1/n}$. Newton's proof by "the method of moments" we should call differentiation, and consisted in showing that if

$$z = \frac{n}{m+n} x^{(m+n)/n}; \quad \text{then} \quad \frac{dz}{dx} = x^{m/n}.$$

It is fair to say also that although he gives Mercator's formula, he gives it as the area under the hyperbola, with no mention of Mercator or of logarithms.

It is time to turn for a moment to the other inventor of the calculus, Gottfried Leibniz. We find him writing in 1677 or 1678 [23],

"In Hyperbol sit $AB = 1$, $BM = x$, $ML = \dfrac{1}{1+x}$,

$$CBMLC = \frac{1}{1}x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots."$$

This is proved by the straight expansion of $1/1+x$, after which there is integration term by term. We find something more interesting a dozen years later, when he writes to Huygens, who is said never to have understood Leibniz calculus of differences [24], "Soit donc x l'abscisse et y l'ordonnée de la courbe, et l'équation comme je vous ay dit

$$\frac{x^3 y}{h} = b^{2xy}.$$

Je désignerai le logarithme de x par log x et nous aurons

$$3 \log x + \log y - \log h = 2xy.$$

supposant que le log de l'unite soit 0 et le log $b = 1$. Donc par la quadrature de

l'hyperbole nous aurons

$$3\int \frac{dx}{x} + \int \frac{dy}{y} - \log h = 2xy$$

$$3\frac{dx}{x} + \frac{dy}{y} = 2xdy + 2ydx,$$

dx sera à dy, on bien DB sera à y comme $2x^2y - x$ est à $3y - 2xy$ c'est à dire DB sera

$$\frac{2x^2y - xa^2}{3a^2 - 2xy}$$

comme vous le demandées, a estant l'unité."

5. Leonhard Euler. It is now time to turn to the man who pulled all this together and who put the number e definitely on the map, Leonhard Euler. This he did in [**25**], beginning in "Caput VII" with the base a. His argument is outlined below:

Since $a^0 = 1$, we may put $a^w = 1 + kw$; $w = \log(1 + kw)$. Assume w to be very small, and write

$$a^{iw} = (1 + kw)^i = 1 + \frac{i}{1}kw + \frac{i(i-1)}{1\cdot 2}k^2w^2 + \frac{i(i-1)(i-2)}{1\cdot 2\cdot 3}k^3w^3 + \cdots.$$

Since w is infinitesimally small, and i is infinitely large, we write $iw = z$

$$a^z = \left(1 - \frac{kz}{i}\right)^i = 1 + kz + \frac{(i-1)}{i\cdot 1\cdot 2}k^2z^2 + \frac{(i-1)(i-2)}{i\cdot 1\cdot 2\cdot 3}k^3k^3 + \cdots.$$

Since i is very large, we may assume $(i-n)/i = 1$, then

$$a^z = 1 + kz + \frac{k^2z^2}{1\cdot 2} + \frac{k^3z^3}{1\cdot 2\cdot 3} + \cdots.$$

If $z = 1$,

$$a = 1 + k + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3}.$$

If we take $a = 10$, the base in the logarithm system of Briggs, Euler gives

$$k = 2.30238, \quad \text{approximately.}$$

For a natural logarithm we take $k = 1$; $a = e$; and

$$e = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots. \tag{8}$$

Euler gives this value to 18 places, without naming the source, namely,

$$e = \lim_{i \to \infty} \left(1 + \frac{1}{i}\right)^i. \tag{9}$$

With regard to the use of the letter e, Euler had long employed it, for we find him writing [26], page 80, "scribitur pro numero cujus logarithmus est unitas e, qui est 2.7182817. . . . " Note that this is Leibniz' b.

I pass to Ch. VII of the *Introductio*. Euler assumes for small values of z,

$$\sin z = z; \qquad \cos z = 1.$$

He then, following DeMoivre, writes,

$$\cos nz = \frac{(\cos z + \sqrt{-1}\sin z)^n + (\cos z - \sqrt{-1}\sin z)^n}{2},$$

$$\sin nz = \frac{(\cos z + \sqrt{-1}\sin z)^n - (\cos z - \sqrt{-1}\sin z)^n}{2\sqrt{-1}}.$$

Putting $nz = v$, and remembering that z is small,

$$\cos v = 1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \cdots,$$

$$\sin v = v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots.$$

Comparing these with the value given previously for a^z, one obtains

$$\cos v = \frac{e^{v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}; \qquad \sin v = \frac{e^{v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}; \tag{10}$$

and

$$v\sqrt{-1} = \log\left[\cos v + \sqrt{-1}\sin v\right]. \tag{11}$$

This last formula was not, strictly speaking, original. Roger Cotes in [27] sought the area of an ellipsoid of revolution. When the rotation is about the minor axis there is no trouble, but when the motion is about the major axis we find him writing "Posset hujus etiam superficiei per Logometriam designari, sed modo inexplicabili . . . arcus erit rationis inter

$$EX + XC\sqrt{-1} \text{ a}CE \text{ mensura ducta in } \sqrt{-1}.$$"

I will leave Euler for a moment to speak of the numerical value of e. William Shanks, who, until quite recently, held the world's record of 707 places for π, had a try at e [28]. Glaisher found an error in this, but Shanks corrected it,

and calculated a value which he was sure was right to 205 places. Glaisher verified 137 of them. Boorman [29] calculated e to 346 places. He acknowledged that he and Shanks agreed only up to 187 places. "One is wrong, which one?" Boorman gives the impression of being a rather amateurish mathematician. Adams [30] calculated $\log_{10} e$ to 272 places, probably all correct. Many years ago I knew a youthful teacher of mathematics who had the vaulting ambition to calculate e by long-hand methods to 1,000 places. I lost sight of him over fifty years ago, probably he died early of heart failure.

I return to Euler. In Caput XVIII of *De Fractionibus continuis* [25], he describes methods of expansion into a continued fraction. When it is a question of turning a rational fraction into a continued one, the process is essentially that of finding a highest common factor, and can be done in only one way. Euler writes

$$e = 2.718281828459 \cdots,$$

$$\frac{e-1}{2} - 0.8591409142295.$$

He writes this in the form,

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{\text{etc.}}}}}}},$$

and remarks [25], page 388, "Cuius fractio ex Calculo infinitesimali dari potest."

Euler assumes that the quotients will increase by 4 each time, so that the fraction goes on indefinitely. Hence e is not a rational fraction.

As for finding this "ex Calculo infinitesimali" he returns to this very much later in life, "Summatio fractionis continuae cujus indices progressionem arithmeticam constituunt" in Vol. 23 of his Opera mentioned in [25]. The method consists in establishing contact with a Riccati differential equation. For a fuller discussion see [31]. Euler did not complete all the details with modern rigor, but what I have just shown is the first attempt to demonstrate the irrationality of e.

We must wait a whole century for anything really new and startling in this line. This came in 1874 with Hermite's proof that e is not an algebraic number [32], that is, not the root of any equation with integral coefficients. A much simpler demonstration is given by Klein in [33].

References

1. Archimedes, Opera omnia. Third ed. Heiberg, vol. III, 1915, p. 79.

2. G. J. Allman, Greek Geometry. Dublin, 1889.

3. Loria, Le Scienze esatte nella antica Grecia. 2nd Ed. Milan, 1914, p. 155.

4. Heath, The Works of Archimedes. Cambridge, 1897, p. 1.

5. Zeuthen, Die Lehre von den Kegelschnitten. Kopenhagen, 1886, p. 463.

6. Die Kegelschnitte des Gregorius a St. Vincento, Abhandlungen zur Geschichte der mathematische Wissenschaften. Vol. XIX, Part 2, Leipzig, 1907.

7. Abraham Gotthilf Kästner, Geschichte der Mathematik. Vol. 3, Göttingen, 1799.

8. Cantor, Geschichte der Mathematik. Vol. 2, Second Ed., Leipzig, 1900, p. 715.

9. Charles Hutton, Mathematical Tables. London, 1804, p. 80.

10 Wallis, Opera Mathematica. Oxford, 1693, Vol. 2.

11. Philosophical Transactions, Abridged. Vol. I, p. 232.

12. Montucla, Histoire des mathématiques. Vol. 2, Paris, 1800, p. 80.

13. Huygens, Oeuvres Complètes. Vol. 14, La Haye, 1920, pp. 433, 441, and 474.

14. Glaisher, The earliest use of the Radix Method for calculating logarithms, Quarterly Journal of Mathematics, Vol. 46, 1914–15, especially p. 174.

15. Tropfke, Geschichte der Elementarmathematik. 3rd Ed., Vol. 2, Berlin, 1933, p. 205.

16. Wallis, Algebra, Ch. XII, Opera, Vol. 2. Oxford, 1693, pp. 57, 58.

17. A most compendius and facile Method for constructing Logarithms exemplified and demonstrated from the Nature of Numbers, Philosophical Transactions, abridged, Vol. IV. 1695–1702. London, 1809, p. 19.

18. Logarithmo-technica Auctore Nicolao Mercatore. See Mazeres, Scriptores Logarithmici, Vol. I. London, 1791, p. 169.

19. Wallis, Logarithmo-technica Nicola Mercatoris, Philosophical Transactions, August, 1668. Mazeres cit. in 18, p. 221.

20. Wallis, Arithmetica Infinitorum. Oxford, 1656. Especially Prop. 59.

21. Newton Commercium Epistolicum Collinsii et aliorum, published by Biot and Leffort, Paris, 1856.

22. An Account of the Book entitled Commercium Epistolicum Collinsii et aliorum. Anonymously by Newton, Philosophical Transactions, Vol. XXIX, London.

23. Leibnizens Mathematische Schriften. Gerhardt Ed., Part 2, Vol. I. Halle, 1858.

24. *Ibid.*, Part I, Vol. 2.

25. Euler, Introductio in Analysin infinitorum. Lausanne, 1748, also his Opera Omnia Series prima, Opera mathematica, Vol. 8.

26. Euler, Meditatio in Experimenta explosione Opera Postuma. Petropoli, 1862.

27. Cotes, Harmonia Mensurarum. Cambridge, 1722, p. 28.

28. Proceedings of the Royal Society, Vol. 6, 1854.

29. Computation of the Naperian Base. Mathematical Magazine, Vol. I, 1884, p. 204.

30. Shanks, On the Modulus of Common Logarithms. Proceedings of the Royal Society, Vol. 43, 1887.

31. Pringsheim, Ueber die ersten Beweise der Irrationalität von e und Sitzungsberichte der K Akademie der Wissenschaften zu München, Vol. 28, 1898.

32. Charles Hermite, Sur la fonction exponentielle. Paris, 1874; Oeuvres, Vol. III, Paris, 1912.

33. Klein, Ausgewählte Fragen der Elementargeometrie. Leipzig, 1895, pp. 47 ff.

"CLOSING-UP" OF SEQUENCE SPACES

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1. Introduction. We consider the linear space $\mathcal{L}$ of sequences of real numbers $(u_1, u_2, \cdots)$ having only a finite number of non-zero terms. On it, we define, for instance, the norm $(\sum_i |u_i|^p)^{1/p}$ where $p>1$. Since the obtained normed linear space is not complete we close it up, that is, imbed it in a complete linear space. That last it is customary to derive either as (1) the Banach space of all those sequences $(u_1, u_2, \cdots)$ having perhaps an infinite number of non-zero terms, for which $(\sum_i |u_i|^p)^{1/p} < +\infty$, or as (2) the Banach space of all fundamental sequences of elements in $\mathcal{L}$. For the above norm both (1) and (2) furnish the space l_p [1, p. 12]. The closures derived by (1) and (2) which obviously depend on the norm on $\mathcal{L}$, may be obtained also for a significant and broad class of norms on $\mathcal{L}$, which, however, do not differ much in character from the norm stated above. It is interesting to note that already for these well-behaving norms the methods (1) and (2) may furnish different closures.

It is the purpose of this note to present these details.

2. Linear spaces. We begin with a brief review of a few elementary notions.

A set $\mathfrak{A}$ of elements $x, y, \cdots$ is termed a linear space if to every ordered couple x, y of $\mathfrak{A}$ corresponds an element of $\mathfrak{A}$ which we term their sum and denote by $x+y$, and to every number α and x in $\mathfrak{A}$ corresponds an element of $\mathfrak{A}$ which we term their product and denote by αx.

Furthermore it is assumed that the operation of addition of elements and the multiplication of numbers by elements are such that the following conditions hold:

(i). $x+y=y+x$.
(ii). $x+(y+z)=(z+y)+z$.
(iii). $x+y=x+z$ implies $y=z$.
(iv). $\alpha(x+y)=\alpha x+\alpha y$.
(v). $(\alpha+\beta)x=\alpha x+\beta x$.
(vi). $\alpha(\beta x)=(\alpha\beta)x$.
(vii). $1x=x$.

It can be verified readily that these conditions imply the existence of a unique element θ in $\mathfrak{A}$ such that $x+\theta=x$ for every x in $\mathfrak{A}$.

By definition, we also put

$$-x = (-1)x; \qquad x - y = x + (-y).$$

A linear space $\mathfrak{A}$ is termed finite dimensional if it contains a finite number of linearly independent elements $x_1, \cdots, x_n$ with the property that every element of $\mathfrak{A}$ can be written as their linear combination. In the last case we term, $x_1, \cdots, x_n$ a basis for $\mathfrak{A}$. If $\mathfrak{A}$ is finite dimensional then every basis for $\mathfrak{A}$ consists of the same number of elements. That number is defined as the dimension of $\mathfrak{A}$.

3. Normed linear spaces; completeness. A linear space $\mathfrak{A}$ is termed normed, if on it is defined a norm, that is, a non-negative function of its elements x which we shall denote by $\|x\|$, such that the following conditions hold:

(1). $\|\theta\| = 0$; $\|x\| > 0$ for $x \neq \theta$.
(2). $\|\alpha x\| = |\alpha| \|x\|$ for any number α.
(3). $\|x+y\| \leqq \|x\| + \|y\|$.

In a normed linear space we define the number

$$\|x - y\|$$

as the distance of the elements x and y.

It is a consequence of (1), (2), (3), that in a normed linear space the distance satisfies the following conditions:

(4). $\|x-y\| > 0$, if $x \neq y$; $\|x-x\| = 0$.
(5). $\|x-y\| = \|y-x\|$.
(6). $\|x-z\| \leqq \|x-y\| + \|y-z\|$.

Thus, a normed linear space is at the same time a metric space.

A sequence $(x_1, x_2, \cdots)$ of elements of a normed linear space $\mathfrak{A}$ is said to be fundamental (or a Cauchy sequence) if,

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \|x_m - x_n\| = 0,$$

while a sequence $(x_1, x_2, \cdots)$ of elements of $\mathfrak{A}$ is said to be convergent to an element x of $\mathfrak{A}$ if,

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Clearly, a sequence of elements which is convergent to an element of the space is always fundamental. The converse is not necessarily true.

A normed linear space where every fundamental sequence of elements of the space is also convergent to an element of that space is termed complete.

A normed linear space which is complete is termed a Banach space.

4. The Cantor-Méray closure of a normed linear space. For a normed linear space $\mathfrak{A}$ its Cantor-Méray closure $[\mathfrak{A}]$ is constructed in the following manner:

Let $[\mathfrak{A}]$ denote the set of all fundamental sequences of elements in $\mathfrak{A}$. For $(x_1, x_2, \cdots)$ and $(y_1, y_2, \cdots)$ in $[\mathfrak{A}]$ and a constant number α we put:

$$(x_1, x_2, \cdots) + (y_1, y_2, \cdots) = (x_1 + y_1, x_2 + y_2, \cdots)$$

and

$$\alpha(x_1, x_2, \cdots) = (\alpha x_1, \alpha x_2, \cdots).$$

It is seen readily that with the above operations of addition of elements and

multiplication of numbers by elements, $[\mathfrak{A}]$ becomes a linear space.

Next we introduce the following conventions:

(a). Two fundamental sequences of elements of $\mathfrak{A}$ are considered identical if and only if the norm of their difference tends towards zero.

(b). The norm of a fundamental sequence of elements in $\mathfrak{A}$ is defined as the limit of the norms of the elements in that sequence.

It is not difficult to see that with the last two conventions $[\mathfrak{A}]$ becomes a normed linear space which is complete [3, p. 106], that is, a Banach space.

Clearly $\mathfrak{A}$ may be considered a linear subspace of $[\mathfrak{A}]$ if for x in $\mathfrak{A}$, $(x, x, \cdots)$ is identified with x. The last identification also preserves the norm.

We express this fact by saying that $\mathfrak{A}$ may be imbedded always in its Cantor-Méray closure $[\mathfrak{A}]$. Furthermore, it is not difficult to prove that $[\mathfrak{A}]$ is the smallest Banach space in which the normed linear space $\mathfrak{A}$ can be imbedded, in the sense that whenever $\mathfrak{A}$ can be imbedded in a Banach space $\mathfrak{B}$ then also $[\mathfrak{A}]$ can be imbedded in $\mathfrak{B}$.

5. Banach spaces and their conjugate spaces. For a given Banach space $\mathfrak{B}$ we are about to define the first and second conjugate space, in symbols $\mathfrak{B}^*$ and $\mathfrak{B}^{**}$.

We say that a functional F is defined on $\mathfrak{B}$ if to every point x in $\mathfrak{B}$ corresponds a number $F(x)$. Functional F is termed additive if $F(x+y)=F(x)+F(y)$ for x, y in $\mathfrak{B}$; F is termed homogeneous if $F(\alpha x)=\alpha F(x)$ for x in $\mathfrak{B}$ and any number α.

An additive and continuous functional F on $\mathfrak{B}$ is termed linear. For additive functionals linearity is equivalent to boundedness [1, p. 54]. We mean hereby that F is linear on $\mathfrak{B}$ if and only if there exists a finite number α such that $|F(x)| \leqq \alpha\|x\|$ for all x in $\mathfrak{B}$.

The least of such numbers α we term the bound of F and denote by $|||F|||$. Thus, by definition:

$$|||F||| = \text{l.u.b.}_{\theta \neq x} \frac{|F(x)|}{\|x\|}.$$

Since a linear functional is necessarily homogeneous, the last expression may be given the following form:

$$|||F||| = \text{l.u.b.}_{\|x\| \leqq 1} |F(x)|.$$

Let $\mathfrak{B}^*$ denote the set of all linear functionals $F, G, \cdots$ on $\mathfrak{B}$. We define addition of elements in $\mathfrak{B}^*$ and multiplication of numbers by elements in $\mathfrak{B}^*$ by means of the obvious relations: $(F+G)(x)=F(x)+G(x)$, $\alpha(F)(x)=F(x)$. Thus, $\mathfrak{B}^*$ becomes a linear space.

It is clear that the bound of a linear functional satisfies the conditions (1), (2), (3) imposed upon a norm. Thus, $\mathfrak{B}^*$ will be normed if for F in $\mathfrak{B}^*$ we put $\|F\| = |||F|||$. Moreover, with the last norm, $\mathfrak{B}^*$ will be complete, that is, a Banach

space. We term $\mathfrak{B}^*$ the conjugate space of $\mathfrak{B}$.

Similarly, for $\mathfrak{B}^*$ we may construct its conjugate space $(\mathfrak{B}^*)^* = \mathfrak{B}^{**}$.

There is a natural relationship combining $\mathfrak{B}$ and $\mathfrak{B}^{**}$ which can be derived by the following reasoning:

Holding x_0 fixed in $\mathfrak{B}$ and varying F in $\mathfrak{B}^*$, $F(x_0)$ represents a linear functional $\mathcal{J}_0$ on $\mathfrak{B}^*$, that is, an element of $\mathfrak{B}^{**}$. Furthermore, $|||\mathcal{J}_0||| = \|x_0\|$.

Thus, the Banach space $\mathfrak{B}$ can be imbedded always in $\mathfrak{B}^{**}$ in a natural way. We express this by writing $\mathfrak{B} \subset \mathfrak{B}^{**}$. The space $\mathfrak{B}$ is termed reflexive whenever in the above imbedding $\mathfrak{B}$ coincides with $\mathfrak{B}^{**}$.

In the future we shall make use of the following:

THEOREM: *A finite dimensional normed linear space is always a reflexive Banach space.*

For a further study of Banach spaces the reader is referred to [1].

6. Symmetric norms on a finite dimensional space.

Definition 1: A real function $\phi = \phi(u_1, \cdots, u_n)$ on the n-dimensional space $\mathfrak{L}_n$ of n tuples $(u_1, \cdots, u_n)$ of real numbers is termed a norm* if it satisfies the following conditions:

(i). $\phi(u_1, \cdots, u_n) > 0$ unless $u_1 = \cdots = u_n = 0$.

(ii). $\phi(\alpha u_1, \cdots, \alpha u_n) = |\alpha| \phi(u_1, \cdots, u_n)$ for any constant α.

(iii). $\phi(u_1 + u_1', \cdots, u_n + u_n') \leqq \phi(u_1, \cdots, u_n) + \phi(u_1', \cdots, u_n')$.

A norm will be termed symmetric if in addition to (i), (ii), (iii), it satisfies the following condition:

(iv). $\phi(u_1, \cdots, u_n) = \phi(\epsilon_1 u_{1'}, \cdots, \epsilon_n u_{n'})$

where $\epsilon_i = \pm 1$ and $1', 2', \cdots n$ denotes any permutation on the first n natural numbers.

To simplify our formulas we shall always assume that the norm also satisfies the following (non-essential for our discussion) condition of "cross-property":

(v). $\phi(1, 0, \cdots, 0) = 1$.

A symmetric norm is a non-decreasing function in the absolute value of each variable u_i; that is, $|u_i| \leqq |u_i'|$ for $i = 1, \cdots, n$ implies $\phi(u_1, \cdots, u_n) \leqq \phi(u_1', \cdots, u_n')$. This is precisely the content of the following:

LEMMA 1. *Let $\phi(u_1, \cdots, u_n)$ denote a symmetric norm on $\mathfrak{L}_n$. Then, for $0 \leqq p_i \leqq 1$ we have,*

$$\phi(p_1 u_1, \cdots, p_n u_n) \leqq \phi(u_1, \cdots, u_n).$$

Proof. By virtue of (iv) we may suppose that all the u_i's are $\geqq 0$. It is clear by induction that it is sufficient to establish the last relation when $0 \leqq p_i \leqq 1$ occurs only for one i, that is,

* Some properties of norms $\phi(u_1, \cdots, u_n)$ on a finite dimensional space have been considered in [2], while symmetric norms (gauge functions) on a finite dimensional space have been discussed in [5].

$$\phi(u_1, \cdots, u_{i-1}, pu_i, u_{i+1}, \cdots, u_n) \leqq \phi(u_1, \cdots, u_{i-1}, u_i, u_{i+1}, \cdots, u_n)$$

for $0 \leqq p \leqq 1$. The last assertion follows from the following simple direct calculation:

$$\begin{aligned}
&\phi(u_1, u_2, \cdots, pu_i, \cdots, u_n) \\
&\quad = \phi\left(\frac{1+p}{2}u_1 + \frac{1-p}{2}u_1, \frac{1+p}{2}u_2 + \frac{1-p}{2}u_2, \cdots, \frac{1+p}{2}u_i\right. \\
&\qquad \left. + \frac{1-p}{2}(-u_i), \cdots, \frac{1+p}{2}u_n + \frac{1-p}{2}u_n\right) \\
&\quad \leqq \phi\left(\frac{1+p}{2}u_1, \frac{1+p}{2}u_2, \cdots, \frac{1+p}{2}u_i, \cdots, \frac{1+p}{2}u_n\right) \\
&\quad + \phi\left(\frac{1-p}{2}u_1, \frac{1-p}{2}u_2, \cdots, \frac{1-p}{2}(-u_i), \cdots, \frac{1-p}{2}u_n\right) \\
&\quad = \frac{1+p}{2}\phi(u_1, u_2, \cdots, u_i, \cdots, u_n) \\
&\qquad + \frac{1-p}{2}\phi(u_1, u_2, \cdots, (-u_i), \cdots, u_n) \\
&\quad = \frac{1+p}{2}\phi(u_1, u_2, \cdots, u_i, \cdots, u_n) \\
&\qquad + \frac{1-p}{2}\phi(u_1, u_2, \cdots, u_i, \cdots, u_n) \\
&\quad = \phi(u_1, u_2, \cdots, u_i, \cdots, u_n).
\end{aligned}$$

Remark. In the proof of our Lemma we have not assumed at all that $\phi(u_1, \cdots, u_n)$ satisfies condition (v) of Definition 1.

LEMMA 2. *Let $\phi(u_1, \cdots, u_n)$ denote a symmetric norm on $\mathfrak{L}_n$. Then,*

$$\max_i |u_i| \leqq \phi(u_1, \cdots, u_n).$$

Proof. Lemma 1, gives

$$\phi(0, \cdots, 0, u_i, 0, \cdots, 0) \leqq \phi(u_1, \cdots, u_{i-1}, u_i, u_{i+1}, \cdots, u_n).$$

By (v) and (ii) of Definition 1 for ϕ the left side of the last inequality equals $|u_i|$, that is,

$$|u_i| \leqq \phi(u_1, \cdots, u_n) \qquad \text{for } i = 1, 2, \cdots, n.$$

This concludes the proof.

LEMMA 3. *For a symmetric norm* $\phi(u_1, \cdots, u_n)$ on $\mathfrak{L}_n$ we have

$$\phi(u_1, \cdots, u_n) \leqq \sum_i |u_i|.$$

Proof. The proof is a simple consequence of conditions (iii), (iv) and (v) for $\phi(u_1, \cdots, u_n)$.

LEMMA 4. *A norm* $\phi(u_1, \cdots, u_n)$ *is a continuous function of the* u_i*'s in the sense that* $\lim_{s\to\infty} u_i^{(s)} = u_i$, $i = 1, 2, \cdots, n$, implies

$$\lim_{s\to\infty} \phi(u_1^{(s)}, \cdots, u_n^{(s)}) = \phi(u_1, \cdots, u_n).$$

Proof. Clearly,

$$|\phi(u_1^{(s)}, \cdots, u_n^{(s)}) - \phi(u_1, \cdots, u_n)|$$

$$\leqq \phi(u_1^{(s)} - u_1, \cdots, u_n^{(s)} - u_n) \leqq \sum_i |u_i^{(s)} - u_i|.$$

This concludes the proof.

We are about to define a conjugate of a given norm $\phi(u_1, \cdots, u_n)$ on $\mathfrak{L}_n$. For a fixed n tuple $(v_1, \cdots, v_n)$ consider the totality of numbers of the form

$$\frac{u_1v_1 + \cdots + u_nv_n}{\phi(u_1, \cdots, u_n)} = f(u_1, \cdots, u_n)$$

taken over the compact (that is, closed and bounded) set of all n tuples, $(u_1, \cdots, u_n)$ for which $|u_1| + \cdots + |u_n| = 1$. Clearly, $f(u_1, \cdots, u_n)$ represents a continuous function on a compact set; hence it assumes its maximum [3, p. 196], which we shall denote by $\psi(v_1, \cdots, v_n)$.

It is seen readily that $\psi(v_1, \cdots, v_n)$ may be defined also as the maximum of the number $f(u_1, \cdots, u_n)$ taken over the set of all n tuples $(u_1, \cdots, u_n) \neq (0, \cdots, 0)$.

The proof of the following two lemmas is immediate:

LEMMA 5. $\psi(v_1, \cdots, v_n)$ *is a norm whenever* $\phi(u_1, \cdots, u_n)$ *is such. If* ϕ *is symmetric, the same holds for* ψ.

LEMMA 6. $u_1v_1 + \cdots + u_nv_n \leqq \phi(u_1, \cdots, u_n)\psi(v_1, \cdots, v_n)$.

The n-dimensional space of n tuples $(u_1, \cdots, u_n)$ on which there is defined the norm $\phi(u_1, \cdots, u_n)$ will be denoted by $\mathfrak{L}_n(\phi)$.

LEMMA 7. *Every linear functional F on $\mathfrak{L}_n(\phi)$ determines an n tuple of numbers $(v_1, \cdots, v_n)$ such that,*

$$F(u_1, \cdots, u_n) = u_1v_1 + \cdots + u_nv_n \qquad \text{for } (u_1, \cdots, u_n) \text{ in } \mathfrak{L}_n(\phi). \tag{1}$$

Furthermore,

$$|||F||| = \text{l.u.b.} \frac{u_1v_1 + \cdots + u_nv_n}{\phi(u_1, \cdots, u_n)}, \tag{2}$$

where the last l.u.b. *is extended over all n tuples $(u_1, \cdots, u_n) \neq (0, \cdots, 0)$.*

CONVERSELY. *Given an n tuple of numbers $(v_1, \cdots, v_n)$, then relationship* (1) *above determines a linear functional F. Its bound is given by* (2).

Proof. Let F be a given linear functional on $\mathfrak{L}_n(\phi)$. We put

$$F(1, 0, \cdots, 0) = v_1, F(0, 1, \cdots, 0) = v_2, \cdots, F(0, 0, \cdots, 1) = v_n.$$

Then, (1) above is a consequence of additivity and homogeneity for F. Clearly, (2) is a consequence of (1) and is the definition of a bound for a linear functional. That the converse holds is immediate. This concludes the proof.

LEMMA 8. *The conjugate space of $\mathfrak{L}_n(\phi)$ may be characterized as $\mathfrak{L}_n(\psi)$.*

Proof. This is a consequence of Lemmas 5 and 7.

The previous considerations justify the following

Definition 2: $\psi(v_1, \cdots, v_n)$ is termed the conjugate norm of the norm $\phi(u_1, \cdots, u_n)$.

LEMMA 9. *Any norm $\phi(u_1, \cdots, u_n)$ on $\mathfrak{L}_n$ is at the same time the conjugate of its conjugate norm $\psi(v_1, \cdots, v_n)$.*

Proof. By Lemma 8, the conjugate space of $\mathfrak{L}_n(\phi)$ may be characterized as $\mathfrak{L}_n(\psi)$. But every finite dimensional space is reflexive. Therefore, the conjugate of $\mathfrak{L}_n(\psi)$ may be characterized as $\mathfrak{L}_n(\phi)$. This concludes the proof.

We conclude our section by pointing out on $\mathfrak{L}_n$ an important class of symmetric norms and their conjugate norms. Our arguments are based on the following well known two inequalities which hold for any pair of n tuples of real numbers $(u_1, \cdots, u_n)$, $(v_1, \cdots, v_n)$.

(**a**). Hölder's inequality:

Let $p>1$ and $1/p+1/q=1$. Then,

$$\left| \sum_{i=1}^{n} u_iv_i \right| \leqq \sum_{i=1}^{n} |u_i||v_i| \leqq \left(\sum_{i=1}^{n} |u_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |v_i|^q \right)^{1/q}.$$

(**b**). Minkowski's inequality:

Let $p>1$. Then,

$$\left(\sum_{i=1}^{n}|u_i+v_i|^p\right)^{1/p}\leqq\left(\sum_{i=1}^{n}|u_i|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|v_i|^p\right)^{1/p}.$$

Due to the significance of these inequalities we sketch their elementary proof.

Proof. (a). Consider the curve $y=x^p$ where $p>1$, $x\geqq 0$. On it we choose a system of n material points with masses $m_1, \cdots, m_n$ and coordinates (x_1, y_1), $\cdots$, (x_n, y_n), respectively. Since our curve is continuously convex downward, the center of gravity (ξ, η) of the system lies above the curve, that is,

$$\eta\geqq\xi^p.$$

Substituting in the last inequality the elementary formulas for the coordinates of the center of gravity, namely,

$$\xi=\frac{\sum_{i=1}^{n}m_ix_i}{\sum_{i=1}^{n}m_i},\qquad \eta=\frac{\sum_{i=1}^{n}m_iy_i}{\sum_{i=1}^{n}m_i},$$

a rearrangement furnishes

$$\left(\sum_{i=1}^{n}m_ix_i\right)\leqq\left(\sum_{i=1}^{n}m_iy_i\right)^{1/p}\left(\sum_{i=1}^{n}m_i\right)^{1/q}.$$

The last inequality holds for arbitrary m_i's and x_i's.

Choosing in it $m_i=|v_i|^q$, $m_ix_i^p=m_iy_i=|u_i|^p$, we get $m_ix_i=|u_i||v_i|$, and hence conclude the proof of Hölder's inequality.

(b). We may assume

$$\sum_{i=1}^{n}|u_i+v_i|^p\neq 0;$$

otherwise the proof is trivial. We have

$$\begin{aligned}\sum_{i=1}^{n}|u_i+v_i|^p&=\sum_{i=1}^{n}|u_i+v_i|^{p-1}|u_i+v_i|\\&\leqq\sum_{i=1}^{n}|u_i+v_i|^{p-1}|u_i|+\sum_{i=1}^{n}|u_i+v_i|^{p-1}|v_i|.\end{aligned}$$

Applying Hölder's inequality to each of the last two sums and observing that $(p-1)q=p$, we get

$$\begin{aligned}\sum_{i=1}^{n}|u_i+v_i|^p\leqq&\left(\sum_{i=1}^{n}|u_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|u_i+v_i|^p\right)^{1/q}\\&+\left(\sum_{i=1}^{n}|v_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|u_i+v_i|^p\right)^{1/q}.\end{aligned}$$

Dividing the last inequality by

$$\left(\sum_{i=1}^{n} |u_i + v_i|^p\right)^{1/q}$$

we conclude the proof of Minkowski's inequality.

It follows therefore that for any $p>1$,

$$\phi(u_1, \cdots, u_n) = \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p}$$

represents a symmetric norm on $\mathfrak{L}_n$. That conditions (i), (ii), (iv), and (v) of Definition 1 are satisfied is clear; (iii) is given by Minkowski's inequality.

To determine the conjugate $\psi(v_1, \cdots, v_n)$ of the last norm we remark that for any non-zero n tuple $(u_1, \cdots, u_n)$, Hölder's inequality furnishes

$$\frac{u_1v_1 + \cdots + u_nv_n}{\left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p}} \leqq \left(\sum_{i=1}^{n} |v_i|^q\right)^{1/q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

It is seen readily that equality is actually assumed when we put $u_i = \epsilon_i |v_i|^{q-1}$ for $i=1, \cdots, n$, where $\epsilon_i = 1$ when $v_i > 0$, and $\epsilon_i = -1$ when $v_i < 0$. Thus, we may express the following statement:

For any $p>1$, the conjugate of the norm

$$\phi(u_1, \cdots, u_n) = \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p}$$

is given by

$$\psi(v_1, \cdots, v_n) = \left(\sum_{i=1}^{n} |v_i|^q\right)^{1/q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

7. Symmetric norms on the linear space of finite sequences. Consider the set whose elements are infinite sequences of real numbers $(u_1, u_2, \cdots)$, having only a finite number of non-zero terms. Defining addition of elements and multiplication of scalars by elements in the obvious fashion we obtain a linear set $\mathfrak{L}$.

Definition 1′: Under a symmetric norm ϕ on $\mathfrak{L}$ we shall understand any function $\phi(u_1, u_2, \cdots)$ defined on $\mathfrak{L}$, subject to conditions (i), $\cdots$, (v) of Definition 1, in which the n tuples are replaced by elements of $\mathfrak{L}$, that is, infinite sequences having only a finite number of non-zero terms.

Clearly, the set of all elements $(u_1, \cdots, u_n, 0, 0, \cdots)$ of $\mathfrak{L}$ with $u_i = 0$ for $i>n$, may be identified with the n-dimensional linear space $\mathfrak{L}_n$.

A given norm $\phi(u_1, u_2, \cdots)$ on $\mathfrak{L}$ defines a norm $\phi_n(u_1, \cdots, u_n) = \phi(u_1, \cdots, u_n, 0, 0, \cdots)$ on each $\mathfrak{L}_n$. Each $\phi_n(u_1, \cdots, u_n)$ determines on $\mathfrak{L}_n$ a conjugate norm $\psi_n(v_1, \cdots, v_n)$.

LEMMA 10. *For a given n tuple* $(v_1, \cdots, v_n)$ *we have*

$$\psi_n(v_1, \cdots, v_n) = \psi_{n+1}(v_1, \cdots, v_n, 0) = \psi_{n+2}(v_1, \cdots, v_n, 0, 0) = \cdots.$$

Their common value we denote by $\psi(v_1, \cdots, v_n, 0, 0, \cdots)$.

Proof. By definition, we have

$$\begin{aligned}\psi_n(v_1, \cdots, v_n) &= \underset{(u_1, \cdots, u_n)}{\text{l.u.b.}} \frac{u_1v_1 + \cdots + u_nv_n}{\phi_n(u_1, \cdots, u_n)} = \underset{(u_1, \cdots, u_n)}{\text{l.u.b.}} \frac{u_1v_1 + \cdots + u_nv_n}{\phi_{n+1}(u_1, \cdots, u_n, 0)} \\ &\leqq \underset{(u_1, \cdots, u_n, u_{n+1})}{\text{l.u.b.}} \frac{u_1v_1 + \cdots + u_nv_n + u_{n+1}0}{\phi_{n+1}(u_1, \cdots, u_n, u_{n+1})} \\ &= \psi_{n+1}(v_1, \cdots, v_n, 0).\end{aligned}$$

To prove the converse inequality, we remark, by Lemma 1, that for any $n+1$ values $u_1, \cdots, u_n, u_{n+1}$ we have

$$\phi_n(u_1, \cdots, u_n) = \phi_{n+1}(u_1, \cdots, u_n, 0) \leqq \phi_{n+1}(u_1, \cdots, u_n, u_{n+1}).$$

Thus,

$$\frac{u_1v_1 + \cdots + u_nv_n + u_{n+1}0}{\phi_{n+1}(u_1, \cdots, u_n, u_{n+1})} \leqq \frac{u_1v_1 + \cdots + u_nv_n}{\phi_n(u_1, \cdots, u_n)}.$$

Definition 2 implies that $\psi_{n+1}(v_1, \cdots, v_n, 0) \leqq \psi_n(v_1, \cdots, v_n)$. Thus,

$$\psi_{n+1}(v_1, \cdots, v_n, 0) = \psi_n(v_1, \cdots, v_n).$$

The rest of the proof is clear.

LEMMA 11. *For a given symmetric norm* $\phi(u_1, u_2, \cdots)$ *on* $\mathfrak{L}$ *the function* $\psi(v_1, v_2, \cdots)$ *represents also a symmetric norm on* $\mathfrak{L}$.

Proof. Clearly, ψ satisfies properties (i), (ii), (iv), and (v). Thus, it is sufficient to verify (iii). Given two sequences representing elements of $\mathfrak{L}$, there exists a number N such that for both sequences the terms with the subscripts greater than N are equal to zero. This means that the sequences may be written in the form $v_1, \cdots, v_N, 0, 0, \cdots$ and $v_1', \cdots, v_N^1, 0, 0, \cdots$. We have

$$\begin{aligned}&\psi(v_1 + v_1', \cdots, v_N + v_N', 0, 0, \cdots) \\ &\qquad = \psi_N(v_1 + v_1', \cdots, v_N + v_N') \\ &\qquad \leqq \psi_N(v_1, \cdots, v_N) + \psi_N(v_1', \cdots, v_N') \\ &\qquad = \psi(v_1, \cdots, v_N, 0, 0, \cdots) + \psi(v_1', \cdots, v_N', 0, 0, \cdots).\end{aligned}$$

This concludes the proof.

Definition 2′: We term $\psi(v_1, v_2, \cdots)$ the conjugate norm of $\phi(u_1, u_2, \cdots)$ on $\mathfrak{L}$.

LEMMA 12. *Any norm* $\phi(u_1, u_2, \cdots)$ *on* $\mathfrak{L}$ *is at the same time the conjugate of its conjugate norm* $\psi(v_1, v_2, \cdots)$ *on* $\mathfrak{L}$.

Proof. Denote by $\bar{\psi}$ the conjugate of ψ. For $(u_1, u_2, \cdots)$ in $\mathfrak{L}$ there exists an N, such that $u_i = 0$ for $i > N$. By definition,

$$\bar{\psi}(u_1, \cdots, u_N, 0, 0, \cdots) = \bar{\psi}_N(u_1, \cdots, u_N)$$

represents the conjugate of $\psi_N(v_1, \cdots, v_N)$. Hence, by Lemma 9, it must be equal to $\phi_N(u_1, \cdots, u_N) = \phi(u_1, \cdots, u_N, 0, 0, \cdots)$.

This concludes the proof.

Once a norm $\phi(u_1, u_2, \cdots)$ is defined on $\mathfrak{L}$ we obtain a normed linear space $\mathfrak{L}(\phi)$ which, in general, will not be complete. In that case $\mathfrak{L}(\phi)$ can always be imbedded in a complete space as outlined by procedure (1) or (2) of the introduction. The Banach spaces furnished by these procedures will in general be different. The details form the content of the following two sections.

8. Strong closure. This term will be attributed to an imbedding method outlined in (1) of the introduction. Its feasibility in the general case is based essentially on Lemma 1, which clearly carries over to norms $\phi(u_1, u_2, \cdots)$ on $\mathfrak{L}$. This Lemma in the required form was stated and proved in the joint paper of John von Neumann and the author [4, p. 568].

Let ϕ be a norm on $\mathfrak{L}$. For a given infinite sequence of real numbers $(u_1, u_2, \cdots)$, of which an infinite number may be different from zero, we form the corresponding sequence of elements: $(u_1, 0, 0, 0, \cdots)$, $(u_1, u_2, 0, 0, \cdots)$, $(u_1, u_2, u_3, 0, \cdots)$, $\cdots$. By Lemma 1, $\phi(u_1, \cdots, u_n, 0, 0, \cdots)$ represents a non-decreasing function of n.

THEOREM 1. *The set of all sequences of real numbers* $(u_1, u_2, \cdots)$ *for which*

$$\lim_{n\to\infty} \phi(u_1, \cdots, u_n, 0, 0, \cdots) < +\infty$$

forms a Banach space $[[\mathfrak{L}(\phi)]]$; *the number on the left side of the last inequality represents the norm of* $(u_1, u_2, \cdots)$.

Proof. Lemma 1 and conditions (ii), (iii) for the norm ϕ prove that the set of sequences $(u_1, u_2, \cdots)$ satisfying the last inequality is a normed linear space. Thus, it is sufficient to prove completeness. Let, therefore, $(u_1^{(i)}, u_2^{(i)}, \cdots)$, $i = 1, 2, \cdots$, form a fundamental sequence of elements of $[[\mathfrak{L}(\phi)]]$. This means that for a given $\epsilon > 0$ we can find a natural $N = N(\epsilon)$, such that

$$\lim_{n\to\infty} \phi(u_1^{(i)} - u_1^{(j)}, \cdots, u_n^{(i)} - u_n^{(j)}, 0, 0, \cdots) < \epsilon$$

whenever both i and j are greater than or at most equal to N. Thus, Lemma 1 gives

(a) $\phi(u_1^{(i)} - u_1^{(j)}, \cdots, u_n^{(i)} - u_n^{(j)}, 0, 0, \cdots) < \epsilon,$

for $i, j \geqq N$; $n = 1, 2, \cdots$.

This, by Lemma 2 implies

$$|u_n^{(i)} - u_n^{(j)}| < \epsilon \quad \text{for } i, j \geqq N; n = 1, 2, \cdots.$$

We put $\lim_{i\to\infty} u_{n_i}^{(i)} = u_n$ for $n = 1, 2, \cdots$. By Lemma 4 and (a) above we get

$$\phi(u_1^{(i)} - u_1, \cdots, u_n^{(i)} - u_n, 0, 0, \cdots) \leqq \epsilon \qquad \text{for } i \geqq N; n = 1, 2, \cdots.$$

Applying again Lemma 1, we obtain

$$\lim_{n\to\infty} \phi(u_1^{(i)} - u_1, \cdots, u_n^{(i)} - u_n, 0, 0, \cdots) \leqq \epsilon \qquad \text{for } i \geqq N.$$

This proves that with $(u_1^{(i)}, u_2^{(i)}, \cdots)$ also $(u_1^{(i)} - u_1, u_2^{(i)} - u_2, \cdots)$ and therefore $(u_1, u_2, \cdots)$ are elements of $[[\mathfrak{L}(\phi)]]$; the sequence $(u_1^{(i)}, u_2^{(i)}, \cdots)$ is convergent in the norm ϕ to $(u_1, u_2, \cdots)$. This concludes the proof.

9. The Cantor-Méray closure. In the present section we shall discuss the quite frequently used method of imbedding any metric space, hence also $\mathfrak{L}(\phi)$, in its Cantor-Méray closure $[\mathfrak{L}(\phi)]$.

THEOREM 2. *The Cantor-Méray closure of $\mathfrak{L}(\phi)$ is always included in its strong-closure. The sequences $(u_1, u_2, \cdots)$ forming the elements of the Cantor-Méray closure of $\mathfrak{L}(\phi)$ must all converge towards zero.*

Proof. Clearly, every element in $\mathfrak{L}(\phi)$ also belongs to $[[\mathfrak{L}(\phi)]]$. Thus, precisely as in the proof of Theorem 1, a fundamental sequence $(u_1^{(i)}, u_2^{(i)}, \cdots)$, $i = 1, 2, \cdots$, of elements of $\mathfrak{L}(\phi)$ determines a sequence $(u_1, u_2, \cdots)$ in $[[\mathfrak{L}(\phi)]]$ to which it converges. This means that for every $\epsilon > 0$, we can find a natural number $N = N(\epsilon)$, such that, for all $i \geqq N$, we have

$$\lim_{n\to\infty} \phi(u_1^{(i)} - u_1, \cdots, u_n^{(i)} - u_n, 0, 0, \cdots) < \epsilon.$$

Thus, again Lemma 1 gives

$$|u_n^{(i)} - u_n| < \epsilon \qquad \text{for } i \geqq N; n = 1, 2, \cdots.$$

In particular

$$|u_n^{(N)} - u_n| < \epsilon \qquad \text{for } n = 1, 2, \cdots.$$

Since $(u_1^{(N)}, u_2^{(N)}, \cdots)$ is an element of $\mathfrak{L}(\phi)$, all its terms, say for $n \geqq n_0$, are equal to zero. Hence

$$|u_n| = |u_n^{(N)} - u_n| < \epsilon \qquad \text{for } n \geqq n_0.$$

Therefore, $\lim_{n\to\infty} u_n = 0$. This concludes the proof.

THEOREM 3. *The strong closure coincides with the Cantor-Méray closure of* $\mathfrak{L}(\phi)$ *if and only if for every sequence of real numbers* $(u_1, u_2, \cdots)$ *the condition*

$$\lim_{n\to\infty} \phi(u_1, \cdots, u_n, 0, 0, \cdots) < +\infty$$

implies

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \phi(u_m, u_{m+1}, \cdots, u_n, 0, 0, \cdots) = 0.$$

Proof. The proof is an immediate consequence of the preceding discussion.

Remark: In particular, when for $(u_1, u_2, \cdots)$ in $\mathfrak{L}$,

$$\phi(u_1, u_2, \cdots) = \left(\sum_i |u_i|^p\right)^{1/p}; \qquad p > 1,$$

the strong closure of $\mathfrak{L}(\phi)$ coincides with its Cantor-Méray closure.

Example. We supplement Theorems 1 and 2 by showing that the Cantor-Méray closure may actually represent a proper subset of the strong closure for $\mathfrak{L}(\phi)$. For this purpose we define

$$\phi(u_1, u_2, \cdots) = \max_i |u_i| \qquad \text{for } (u_1, u_2, \cdots) \text{ in } \mathfrak{L}.$$

Clearly, ϕ is a norm. It is seen readily that the strong closure of $\mathfrak{L}(\phi)$ represents the Banach space of all bounded sequences $(u_1, u_2, \cdots)$ of real numbers, where l.u.b. $|u_i|$ represents the norm [1, p. 11]. The Cantor-Méray closure of $\mathfrak{L}(\phi)$ represents the Banach space of all sequences of real numbers converging towards 0, with the least upper bound stated above as its norm [1, p. 181].

10. Strong closures as conjugate spaces of Cantor-Méray closures. In the present section we indicate another relationship between the strong and Cantor-Méray closure of $\mathfrak{L}(\phi)$.

THEOREM 4. *Let* $\phi(u_1, u_2, \cdots)$ *denote a norm on* $\mathfrak{L}$ *and* $\psi(v_1, v_2, \cdots)$ *its conjugate norm. Then, the conjugate space of the Cantor-Méray closure of* $\mathfrak{L}(\phi)$ *may be characterized as the space of all those sequences of real number* $(v_1, v_2, \cdots)$ *(of which an infinite number of* v_i*'s may be* $\neq 0$*) for which*

$$\lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots) < +\infty.$$

The number on the left side of the last inequality represents the norm of $(v_1, v_2, \cdots)$.

Thus, the conjugate space of the Cantor-Méray closure of $\mathfrak{L}(\phi)$ *may be interpreted as the strong closure of* $\mathfrak{L}(\psi)$. *Conversely, the strong closure of* $\mathfrak{L}(\phi)$ *may be interpreted as the conjugate space of the Cantor-Méray closure of* $\mathfrak{L}(\psi)$.

Proof. Clearly, a linear functional F on $\mathfrak{L}(\phi)$ can be extended in a unique manner without changing the value of its bound to a linear functional on $[\mathfrak{L}(\phi)]$. Thus, the conjugate space of $[\mathfrak{L}(\phi)]$ may be identified with the space of linear functionals on $\mathfrak{L}(\phi)$.

So let F denote a linear functional on $\mathfrak{L}(\phi)$ and $|||F|||$ represent its bound. As for the finite-dimensional case we put $F(1, 0, 0,0, \cdots) = v_1$, $F(0, 1, 0, 0 \cdots) = v_2$, $F(0, 0, 1, 0, \cdots) = v_3, \cdots$. Thus, F determines a sequence of numbers $(v_1, v_2, \cdots)$. Clearly,

$$\frac{u_1v_1 + \cdots + u_nv_n}{\phi_n(u_1, \cdots, u_n)} = \frac{F(u_1, \cdots, u_n, 0, 0, \cdots)}{\phi(u_1, \cdots, u_n, 0, 0, \cdots)} \leqq |||F|||.$$

The last inequality holds for any n tuple $(u_1, \cdots, u_n)$. Thus,

$$\psi(v_1, \cdots, v_n, 0, 0, \cdots) = \psi_n(v_1, \cdots, v_n) \leqq |||F|||.$$

By Lemma 1, $\psi(v_1, \cdots, v_n, 0, 0, \cdots)$ is a nondecreasing function of n. Thus,

$$\lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots) \leqq |||F|||.$$

On the other hand, using Lemma 6 for an element $(u_1, \cdots, u_n, 0, 0, \cdots)$ in $\mathfrak{L}(\phi)$, we have

$$\begin{aligned}\frac{F(u_1, \cdots, u_n, 0, 0, \cdots)}{\phi(u_1, \cdots, u_n, 0, 0, \cdots)} = \frac{u_1v_1 + \cdots + u_nv_n}{\phi_n(u_1, \cdots, u_n)} &\leqq \frac{\phi_n(u_1, \cdots, u_n)\psi_n(v_1, \cdots, v_n)}{\phi_n(u_1, \cdots, u_n)} \\ &= \psi_n(v_1, \cdots, v_n) = \psi(v_1, \cdots, v_n, 0, 0, \cdots) \\ &\leqq \lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots).\end{aligned}$$

Thus,

$$|||F||| \leqq \lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots).$$

We have shown, therefore, so far, that a linear functional F on $\mathfrak{L}(\phi)$ determines a unique sequence $(v_1, v_2, \cdots)$ of real numbers, for which

$$|||F||| = \lim_{n\to\infty} \psi(v_1, \cdots, v_n, 0, 0, \cdots) < +\infty.$$

Conversely, it is readily seen that every sequence of numbers $(v_1, v_2, \cdots)$, for which the limit on the left side of the last inequality is finite, corresponds to some linear functional F on $\mathfrak{L}(\phi)$.

This proves the first part of our theorem. Using this and Lemma 12 we establish the remaining part. This concludes the proof.

References

1. S. Banach: Théorie des Opérations Linéaires, Warsaw, 1932.
2. T. Bonnesen und W. Fenchel: Theorie der konvexen Körper, Springer, Berlin, 1934.
3. F. Hausdorff: Mengenlehre, Göschen, Berlin, 1935.
4. Robert Schatten and John von Neumann: The Cross-Space of Linear Transformations III, Annals of Math., vol. 49, 1948, pp. 557–582.
5. John von Neumann: Some Matrix-Inequalities and Metrization of Matrix-Space, Tomsk University Review, vol. I, 1937, pp. 286–300.

THE BEST POLYNOMIAL APPROXIMATION OF FUNCTIONS

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1. Introduction. One of the most interesting mathematical studies is that of finding the "best" polynomial approximation of a given function. The solution of the problem depends, of course, on the meaning we attach to the expression "the best polynomial approximation." In this expository paper we will discuss some of the aspects of best approximation in the classical Tchebycheff sense.

2. Statement of the problem. Let $f(x)$ and $\theta(x)$ be given functions which are continuous in the finite closed interval (a, b). The problem is to find values of the parameters $p_0, p_1, \cdots, p_n$ such that the polynomial $P_n(x) = p_0x^n + p_1x^{n-1} + \cdots + p_n$, of degree not exceeding n, will differ in absolute value as little as possible from $f(x)$ in the sense that the deviation

$$D_n(p) = D_n(p_0, p_1, \cdots, p_n) = \max_{a \leqq x \leqq b} \left| f(x) - P_n(x)/\theta(x) \right| , \tag{1}$$

shall be as small as possible for all x in (a, b). If $p_0, p_1, \cdots, p_n$ are values of the parameters which give $D_n(p)$ its minimum value then we call the function

$$\phi_n(x; p) = f(x) - P_n(x)/\theta(x) \tag{2}$$

a function of least deviation from zero corresponding to all functions of the given type $f(x)$. Let $x_1 < x_2 < \cdots < x_m$, $m \geqq 1$, $(x_i = x_{i,n})$ be the points in (a, b) where $\phi_n(x; p)$ attains its extreme value $\pm D_n(p)$. That is, $\phi_n(x_i; p) = \epsilon_i D_n(p)$, $\epsilon_i = \pm 1$, $(i = 1, 2, \cdots, m)$. We call each x_i a point of deviation of $\phi_n(x; p)$.

3. Fundamental theorems. The following basic theorems are necessary for the discussion which follows. Limitations of space forbid the presentation of proofs. For these the reader may consult [11]. At the outset we assume that a function of least deviation from zero exists, and that it is unique in the sense that there cannot be two sets of parameters which correspond to two different $\phi_n(x; p)$-functions having the same minimum deviation.

(i) The function $\phi_n(x; p)$ has at least $n+2$ points of deviation $x_1 < x_2 < \cdots < x_{n+2}$; $\phi_n(x_i; p) = \epsilon_i D_n(p)$, $\epsilon_i = \pm 1$, $(i = 1, 2, \cdots, n+2)$.

(ii) If the parameters $p_0, p_1, \cdots, p_n$ are so chosen that the sequence $\epsilon_i D_n(p)$, $(i = 1, 2, \cdots, m)$, has at least $n+1$ variations of signs, then the corresponding function $\phi_n(x; p)$ is a function of least deviation.

4. Applications. We will now solve two typical problems in the theory of best approximation.

Problem 1. In (2) take $\theta(x) = 1$, $f(x) = -x^{n+1}$, and seek the polynomial $F(x) = x^{n+1} + p_0x^n + \cdots + p_n$ which deviates the least from zero in $(-1, 1)$.

Solution. By *(i)*, $F(x)$ has at least $n+2$ points of deviation. The only possible points of deviation are the end points $x = \pm 1$ and the points within $(-1, 1)$ at which $F'(x) = 0$. But $F'(x) = 0$ has at most n roots, hence $F(x)$ has exactly $n+2$ points of deviation, namely, the points $x = 1$, $x = -1$, and the n roots of $F'(x) = 0$.

We denote the latter by $x_1, x_2, \cdots, x_n$, and write D for $D_n(p)$. Clearly the points $x_1, x_2, \cdots, x_n$ are double roots of $F(x)^2 - D^2 = 0$, so that we may write

$$F(x)^2 - D^2 = c(x-1)(x+1)\prod_{i=1}^{n}(x-x_i)^2. \tag{3}$$

Since $F(x) = x^{n+1} \cdots$ we see at once that $c = 1$. Again

$$F'(x) = c_1(x-x_1)(x-x_2)\cdots(x-x_n) = (n+1)x^n + \cdots$$

so that $c_1 = n+1$. Thus (3) may be written in the form

$$F(x)^2 - D^2 = (x^2-1)[F'(x)/(n+1)]^2,$$

or, as a differential equation,

$$\frac{F'(x)dx}{[F(x)^2 - D^2]^{1/2}} = \frac{(n+1)dx}{(x^2-1)^{1/2}}.$$

Integrating we obtain

$$F(x) + [F(x)^2 - D^2]^{1/2} = C[x + (x^2-1)^{1/2}]^{n+1}, \tag{4}$$

and hence

$$F(x) - [F(x)^2 - D^2]^{1/2} = D^2C^{-1}[x - (x^2-1)^{1/2}]^{n+1}. \tag{5}$$

To determine C notice that $F(1) = \pm D$, and that $x = 1$ in (4) gives $C = \pm D$, with the same sign as $F(1)$. Substituting this value of C into (4) and (5), and adding, we obtain

$$F(x) = \pm 2^{-1}D\{[x + (x^2-1)^{1/2}]^{n+1} + [x - (x^2-1)^{1/2}]^{n+1}\}. \tag{6}$$

Equation (6) enables us to evaluate D. Expanding the right-hand side in powers of x, and recalling that $F(x) = x^{n+1} + \cdots$, we conclude that $D = 2^{-n}$. If we now let $x = \cos\phi$ in (6), we get

$$F(x) = D\cos((n+1)\phi) = 2^{-(n+1)}\cos((n+1)\arccos x). \tag{7}$$

Notice that as x varies from -1 to $+1$, $D\cos((n+1)\phi)$ passes alternately through the values $D, -D, D, \cdots$ with $n+1$ variations of signs, hence, by (*ii*) we conclude that this trigonometric polynomial is the required function of least deviation from zero.

The transformation $2x = (b+a) + (b-a)t$, $-1 \leqq t \leqq 1$, $a \leqq x \leqq b$, applied to (7) shows immediately that the polynomial

$$2\left(\frac{b-a}{4}\right)^{n+1}\cos\left((n+1)\arccos\left(\frac{2x-b-a}{b-a}\right)\right) = x^{n+1} + \cdots$$

deviates the least from zero on (a, b) among all polynomials of the form, $x^{n+1} + \cdots$, a conclusion which is often used in the general theory of polynomials. Tchebycheff [9] used this and similar results to prove many beautiful and useful theorems in the theory of equations.

Problem 2. (Poncelet's problem.) Our second illustration is the celebrated problem of Poncelet which in the masterly hands of Tchebycheff became the cornerstone of the theory of best approximation. Poncelet sought an approximation of $(x^2+y^2)^{1/2}$ by a linear function $\alpha x+\beta y$ such that the maximum of the relative error

$$\left| \frac{\alpha x + \beta y}{(x^2 + y^2)^{1/2}} - 1 \right|$$

should be as small as possible when $ax \leqq y \leqq bx$, $0 \leqq a < b$, $x > 0$. If we set $z = y/x$, then the problem is to determine α and β so that the deviation D of the function $\phi(z) = (\alpha+\beta z)/(1+z^2)^{1/2} - 1$ shall be a minimum in $a \leqq z \leqq b$. This is a problem of the type we have been studying with $\theta(x) = (1+z^2)^{1/2}$, $f(z) = 1$.

Exactly as in Problem 1, the points of deviation are $z = a$, $z = b$, and $z = \beta/\alpha$, the point at which $\phi'(z) = 0$. Since there must be at least two variations of signs, we write

$$\phi(a) = \pm D, \tag{8}$$

$$\phi(\beta/\alpha) = \mp D, \tag{9}$$

$$\phi(b) = \pm D. \tag{10}$$

These three equations enable us to determine α, β, and D. To this end, introduce $a = \tan \delta_0$, $b = \tan \delta_1$, then from (8) and (10) we have

$$\alpha \cos \delta_0 + \beta \sin \delta_0 - 1 = \pm D, \tag{11}$$

$$\alpha \cos \delta_1 + \beta \sin \delta_1 - 1 = \pm D; \tag{12}$$

hence, by subtraction,

$$\alpha \sin \frac{\delta_0 + \delta_1}{2} = \beta \cos \frac{\delta_0 + \delta_1}{2},$$

so that

$$\alpha = C \cos \frac{\delta_0 + \delta_1}{2}, \qquad \beta = C \sin \frac{\delta_0 + \delta_1}{2},$$

where $C \neq 0$ is a constant to be determined. Substituting these values of α and β into (12) and (9) we obtain, in turn,

$$C \cos \frac{\delta_0 - \delta_1}{2} - 1 = \pm D, \qquad C - 1 = \mp D,$$

and, by addition, we find that

$$C = \sec^2 \frac{\delta_0 - \delta_1}{4},$$

so that $C>1$. Hence $D=C-1=\sec^2(\delta_0-\delta_1)/4-1=\tan^2(\delta_0-\delta_1)/4$, and $\phi(a)=-D, \phi(\beta/\alpha)=+D, \phi(b)=-D$. Thus by (*ii*), we have discovered the required function of least deviation and the problem is completely solved.

It remains to give a numerical example. Suppose, for instance, that we take $0\leqq y\leqq x$ so that $a=0$, $b=1$, $\delta_0=0°$, $\delta_1=45°$. Then $\alpha=0.9605$, $\beta=0.3978$, and $D=0.0396$. Hence

$$(x^2+y^2)^{1/2}=0.9605x+0.3978y, \qquad x\geqq y\geqq 0,$$

with a relative error less than 4%.

The following result may be derived in a similar manner.

$$(x^2+y^2+z^2)^{1/2}\cong 0.940x+0.328y+0.299z, \quad x\geqq y\geqq z\geqq 0,$$

with a relative error less than 6%. We may prove that

$$e^x\cong 0.8459x+0.9549x+1.0007, \text{ error}<0.088, \qquad 0\leqq x\leqq 1;$$

$$e^{(x+y)/2}\cong 0.946+0.649(x+y)+0.421xy, \text{ error}<0.054, \qquad 0\leqq x, y\leqq 1;$$

$$e^{(x+y)/2}\cong 0.908+0.638x+1.050y, \text{ error}<0.054, \qquad 0\leqq x, y\leqq 1;$$

$$|x|\cong 0.0675+1.9313x^2-1.0665x^4, \text{ error}<0.0677, \qquad -1\leqq x\leqq 1.$$

5. Conclusion. The problem of Poncelet stands at the beginning of the theory of best approximation. G. Zolotareff [12], W. Markoff [5], S. Bernstein [2], to mention only a few names, have greatly extended the theory. Present day mathematicians are still finding this a fertile field for further research. We mention, for example, the work of N. Achyeser [1] on polynomials of least deviation in two disjoint intervals, and that of J. Shohat [8] who uses the close connection between the theory of interpolation and that of best approximation. A problem similar to Poncelet's problem was solved by J. V. Uspensky [10]. As examples of problems in a closely related field we list the work of I. Schur [7], and that of P. Erdös and G. Szegö [4]. There is still much to be done in extending the theory to functions of several variables. In 1889 Tchebycheff studied rational approximations of the function $x^{-1/2}$ and applied his theory to the approximate evaluation of elliptic integrals. The theory of best approximation can be used to give approximate solutions of differential equations. The recent book by E. Remes [6] is recommended for its success in applying the classical theory of best approximation to practical engineering problems.

References

1. N. Achyeser, Über einige Funktionen, welche in zwei gegebenen Intervallen am wenigsten von Null abweichen, Bulletin de l'Académie des Sciences de l'URSS, series 7, 1932, pp. 1163–1202.

2. S. Bernstein, Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réele, Paris, 1926.

3. E. Borel, Leçons sur les Fonctions des Variables Réelles et les Développements en Séries de Polynomes, Paris, 1905.

4. P. Erdös and G. Szegö, On a Problem of I. Schur, Annals of Mathematics, vol. 43, 1942, pp. 451–470.

5. W. Markoff, On Functions Deviating the Least from Zero on a Given Interval (in Russian), 1892; German translation, Mathematische Annalen, vol. 77, 1916, pp. 213–255.

6. E. Remes, On Methods for Obtaining the Best Approximation of Functions, in the Sense of Tchebycheff (in Ukrainian). Math. Institute of the Ukrainian Academy of Science, Kieff, 1935.

7. I. Schur, Über das Maximum des absoluten Betrages eines Polynoms in einem gegebenen Intervall, Mathematische Zeitschrift, vol. 4, 1919, pp. 271–287.

8. J. Shohat, The Best Polynomial Approximation of Functions Possessing Derivatives, Duke Mathematical Journal, vol. 8, 1941, pp. 376–385.

9. P. Tchebycheff, Collected Papers, vol. 2, no. 11, pp. 189–215.

10. J. V. Uspensky, A Minimum Problem, American Mathematical Monthly, vol. 40, 1933, pp. 5–10.

11. Ch.-J. de la Vallée-Poussin, Leçons sur l'Approximation des Fonctions d'une Variable Réelle, Paris, 1919.

12. G. Zolotareff, On a question concerning a minimum value (in Russian), Oeuvres, vol. 2, 1902, pp. 130–166; Applications of elliptic functions to questions concerning functions which deviate the least from zero (in Russian), Oeuvres, vol. 2, pp. 1–59. See also Oeuvres, vol. 1, pp 369–374.

MATHEMATICAL NOTES

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ON THE METHOD OF CHARACTERISTICS*

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1. Introduction. In problems in applied mathematics, it is frequently necessary to consider systems of equations which contain both finite and differential relations.[1] As an example, let us consider a set of $m+n$ unknown functions which are assumed to satisfy n first order quasi-linear partial differential equations and m finite relations. The straightforward method of solution (method A) is to solve the finite relations for m of the variables, substitute in the differential equations, and solve the resulting set of n first order equations in n unknowns. However, it may prove more convenient to differentiate each of the finite relations with repect to one of the independent variables and solve the resulting set of $m+n$ first-order equations in $m+n$ unknowns (method B).

* The results presented in this paper were obtained in the course of research conducted under Contract N7onr-358, sponsored jointly by the Office of Naval Research and the Bureau of Ships at Brown University.

[1] See, for example, P. S. Symonds, "On the general equations of problems of axial symmetry in the theory of plasticity," Q. Appl. Math. 6, 448–452 (1949). It was a consideration of this problem which suggested the present note.

Since a system of first order equations is often most easily solved by the method of characteristics, the question arises as to the relation between the characteristics obtained by the two methods. It is intuitively plausible that the $m+n$ families of characteristics obtained by method B will consist of the n families obtained by method A, plus families of total multiplicity m parallel to one or the other coordinate axis. In the present note, a proof of this statement will be given.

2. Discussion. Let the given set of equations be

$$\sum_{j=1}^{n}\left[A_{ij}\frac{\partial u_j}{\partial x}+B_{ij}\frac{\partial u_j}{\partial y}\right]+\sum_{\alpha=1}^{m}\left[C_{i\alpha}\frac{\partial v_j}{\partial x}+D_{i\alpha}\frac{\partial v_j}{\partial y}\right]=E_i, \tag{1}$$

$$v_\alpha = f_\alpha(x, y, u_1, \cdots, u_n)\ ;$$

here Latin indices range from 1 to n, and Greek indices from 1 to m. The functions $A_{ij}, \cdots, E_i$ may depend upon x, y, u_j, v_α, but are independent of the derivatives of u_j and v_α. In order to save space, we shall consider the case where $n=2$, $m=1$, although the method is perfectly general. In this case (1) consists of three equations of the form

$$A_1u_x + B_1u_y + C_1v_x + D_1v_y + E_1w_x + F_1w_y = G_1, \tag{2}$$

$$A_2u_x + B_2u_y + C_2v_x + D_2v_y + E_2w_x + F_2w_y = G_2,$$

$$w = f(x, y, u, v), \tag{3}$$

where the subscripts x and y indicate differentiation.

To solve the problem by method A, we substitute (3) into (2), obtaining

$$(A_k + E_kf_u)u_x + (B_k + F_kf_u)u_y + (C_k + E_kf_v)v_x + (D_k + F_kf_v)v_y = G_k - E_kf_x - F_kf_y, \qquad k = 1, 2. \tag{4}$$

The characteristics of (4) are those curves along which the determinant

$$\Delta_A = \begin{vmatrix} A_1 + E_1f_u & B_1 + F_1f_u & C_1 + E_1f_v & D_1 + F_1f_v \\ A_2 + E_2f_u & B_2 + F_2f_u & C_2 + E_2f_v & D_2 + F_2f_v \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix}$$

vanishes.

To solve the problem by method B we differentiate (3) with respect to x, obtaining

$$f_uu_x + f_vv_x - w_x = f_x. \tag{5}$$

The characteristics of the three equations (2) and (5) are the curves along which the determinant

$$(6)\qquad \Delta_B = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 & E_1 & F_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 & F_2 \\ f_u & 0 & f_v & 0 & -1 & 0 \\ dx & dy & 0 & 0 & 0 & 0 \\ 0 & 0 & dx & dy & 0 & 0 \\ 0 & 0 & 0 & 0 & dx & dy \end{vmatrix}$$

vanishes. We perform elementary operations on the columns and rows respectively to obtain

$$(7)\qquad \Delta_B = \left| C_1 + f_u C_5 \quad C_2 + f_u C_6 \quad C_3 + f_v C_5 \quad C_4 + f_v C_6 \quad C_5 \quad C_6 \right|,$$

$$(8)\qquad \Delta_B = \left| R_1 \quad R_2 \quad R_3 \quad R_4 \quad R_5 \quad R_6 - f_u R_4 - f_v R_5 \right|;$$

here C_i represents the ith column of (6), and R_j the jth column of (7). Finally we may expand (8) in terms of the cofactors of the third and last rows respectively to obtain

$$\Delta_B = (-1)(dy)(\Delta_A).$$

Therefore, the characteristics according to method B are the four families of curves obtained as the characteristics by method A, plus the family of curves $dy=0$.

3. Conclusion. In the general case, an analogous procedure applied to (1) leads to the relation

$$\Delta_B = \pm (dy)^p (dx)^{m-p} (\Delta_A);$$

here p represents the number of finite relations differentiated with respect to x, and $m-p$ the number differentiated with respect to y.

The same conclusion can be drawn in the case where some of the equations are of the second or higher order. Thus, in general, it is permissible to differentiate any or all of a set of equations before obtaining the characteristics.

A REMARK ON INFINITE GROUPS

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1. Introduction. L. J. Paige [1] has shown that if G is a finite abelian group such that the number of elements of order two is different from unity, then there exists a one-to-one mapping $x \to \theta(x)$ of G onto itself such that $x \to \eta(x) = x \cdot \theta(x)$ is also a one-to-one mapping of G onto itself. This result was essentially proved also by Vijayaraghavan and Chowla [2] (independently of Paige) by explicitly constructing such mappings. Later Paige in effect proved (cf. [3], pp. 50–51) that mappings θ exist for finitely generated infinite abelian groups in which the number of elements of order two is different from unity. It is the purpose of this note to remark that the well-ordering principle implies immediately

the existence of a mapping θ for any infinite group whatever, in fact for any infinite quasigroup. (A quasigroup is a set G in which is defined a product xy for every ordered pair x, y of elements of G such that if, in the equation $xy=z$, any two of the symbols x, y, z are assigned as elements of G, the third is uniquely determined as an element of G.)

2. Theorem. We shall establish the following result.

THEOREM. *For any infinite quasigroup G there exists a one-to-one mapping $x\to\theta(x)$ of G onto itself such that $x\to\eta(x)=x\cdot\theta(x)$ is also a one-to-one mapping of G onto itself. Moreover if a and b are any given elements of G, we can choose θ so that $\theta(a)=b$.*

3. Proof. We may assume that $G=\{g\alpha\}$, where α runs over the ordinal numbers less than some fixed initial ordinal δ (*cf.* [4], pp. 71–72). We want to show that for each α less than δ we can define x_α, $y_\alpha=\theta(x_\alpha)$, and $z_\alpha=\eta(x_\alpha)$, such that $x_\alpha y_\alpha=z_\alpha$, and such that each element of G occurs once and only once in $\{x_\alpha\}$, in $\{y_\alpha\}$, and in $\{z_\alpha\}$. First we choose $x_1=a$, $y_1=b$, $z_1=ab$. Then suppose β is any ordinal less than δ and assume that we have defined x_α, y_α, z_α for $\alpha<\beta$. Put $X_\beta=\{x_\alpha\}_{\alpha<\beta}$, $Y_\beta=\{y_\alpha\}_{\alpha<\beta}$, $Z_\beta=\{z_\alpha\}_{\alpha<\beta}$. Suppose t_β is the first element of G not in $X_\beta\cap Y_\beta\cap Z_\beta$. If $t_\beta\notin X_\beta$, we choose x_β as t_β and y_β as the first element of G not in $Y_\beta\cup\{y\,|\,t_\beta y\in Z_\beta\}$. (The set $Y_\beta\cup\{y\,|\,t_\beta y\in Z_\beta\}$ is a proper subset of G since δ is an initial ordinal.) If $t_\beta\in X_\beta$ but $t_\beta\notin Y_\beta$, we choose y_β as t_β and z_β as the first element of G not in $Z_\beta\cup\{z\,|\,z\in X_\beta t_\beta\}$. If $t_\beta\in X_\beta$ and $t_\beta\in Y_\beta$ but $t_\beta\notin Z_\beta$, we choose z_β as t_β and x_β as the first element of G not in $X_\beta\cup\{x\,|\,t_\beta\in x Y_\beta\}$. This completes the proof.

4. Remarks. It is known (cf. [3], pp. 40–42 and the first part of the proof of Theorem II.3 on p. 50) that if G is an abelian group, then the existence of a mapping θ (of the above type) with $\theta(1)=1$ ensures the existence of a neofield for which G is the multiplicative group. Thus as a corollary of the above theorem we see that any infinite abelian group can be the multiplicative group of a neofield. Our theorem also implies (cf. [1], §1) that (i) every infinite quasigroup is isotopic to an idempotent quasigroup, (ii) every infinite group is a subgroup of the automorphism group of some loop.

The question of the existence of mappings θ for finite non-abelian groups appears difficult.

References

1. L. J. Paige, A note on finite abelian groups, Bull. Amer. Math. Soc., vol. 53 (1947), pp. 590–593.

2. T. Vijayaraghavan and S. Chowla, On complete residue sets, Quart. J. Math. (Oxford Ser.), vol. 19 (1948), pp. 193–199.

3. L. J. Paige, Neofields, Duke Math. J., vol. 16 (1949), pp. 39–60.

4. F. Hausdorff, Mengenlehre, third edition.

A NOTE ON SUMMATION

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The use of arithmetic means in summing nonconvergent series suggests that perhaps the harmonic and geometric means may yield results of interest. Proceeding in a manner similar to that of Hölder, we consider the series

$$U = \sum_{k=1}^{\infty} u_k. \quad \text{Let } s_n = \sum_{k=1}^{n} u_k, \quad \text{and let } {}_1H_n = n \Big/ \sum_{k=1}^{n} (1/s_k).$$

Then ${}_1H_n$ is merely the harmonic mean of the first n partial sums of U. In general, let ${}_rH_n = n/\sum_{k=1}^{n}(1/{}_{r-1}H_k)$. If r is the smallest integer for which $\lim_{n\to\infty} {}_rH_n$ exists, we shall call the series summable (r, H) by harmonic means* of order r. This, of course, is not a regular method of summation, for it is not applicable to series having any partial sum zero, nor to series converging to zero.

As an example of a series summable $(1, H)$, consider the series whose partial sums are recurrently $2/3, 1, 2$. Then $\lim_{n\to\infty} {}_1H_n = 1$. Hence the series is summable $(1, H)$ to the value 1. Similarly, we can easily construct a series for which the harmonic means of the partial sums are recurrently $2/3, 1, 2$. That is, for six terms, $U = 2/3 + 4/3 - 4 + 20/9 - 11/9 + 1/2 + \cdots$. Then $s_1 = 2/3$, $s_2 = 2$, $s_3 = -2$, $s_4 = 2/9$, $s_5 = -1$, $s_6 = -1/2$, $\cdots$, and ${}_1H_{3n} = 2/3$, ${}_1H_{3n+1} = 1$, ${}_1H_{3n+2} = 2$ for $n = 1, 2, \cdots$. Clearly, a series constructed in such a manner is summable $(2, H)$ to the value 1.

The harmonic mean has the desired summation property that if U is a convergent series with the ordinary sum S, satisfying the special conditions above, then U is summable $(1, H)$ to S. For let $p_n = 1/s_n$ and $P = 1/S$. Then $\lim_{n\to\infty} p_n = P$. The p_n may be considered as the partial sums of a convergent series. Hence, $\lim_{n\to\infty} (\sum_{k=1}^{n} p_k)/n = P$, or $\lim_{n\to\infty} n/\sum_{k=1}^{n}(1/s_k) = S$.

A series may be summable (r, H) when it is not summable (C, r), and the converse is true. Consider the series whose partial sums occur in pairs, either 1, 1 or 10, 10/19. The harmonic mean of either pair is 1. The arithmetic mean of the first pair is 1 and that of the second is 100/19. Let us generate a series in the following manner: For the first pair of partial sums, take 1, 1. Then take 10, 10/19 enough times so that the arithmetic mean exceeds 4. Then take 1, 1 enough times so that the arithmetic mean is less than 2. If this process is continued, we obtain a series that is summable $(1, H)$ to the value 1 but is not summable $(C, 1)$. Obviously, a similar series could be constructed that would be summable $(C, 1)$ but not $(1, H)$.

In view of the arithmetic-geometric-harmonic mean inequality, it is apparent that the geometric mean of the partial sums is also applicable to series having only positive partial sums; and so, indeed, are means of other orders.

* Not to be confused with

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n} \frac{s_{n-k}}{k+1}. \quad \text{Ed.}$$

CLASSROOM NOTES

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$C_{n,r}$, WHERE THE n THINGS INCLUDE IDENTITIES

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1. Introduction. Given the set of objects:

$$A,\ A,\ A,\ A,\ B,\ B,\ B,\ C,\ C,\ D,$$

in which the first 4 are identical, the next three, and so forth, it is desired to find the number of different combinations of those 10 things taken 5 at a time. This number is quite different from $C_{10,5}$, where the 10 things are considered to be all different.

2. Notation. $[a, b, c, \cdots]$ is a finite set of objects having a subset of a identical objects, a second subset of b identical objects, and so forth. The set mentioned in Section 1 would be $[4, 3, 2, 1]$, since there are 4 identical A's making up the first subset, 3 B's in the second, and so forth.

The number of objects in a set is always called n.

$$n = a + b + c + \cdots . \tag{1}$$

All the letters used are supposed to be non-negative integers.

If two or more of the subsets of identical objects have the same number of elements, the notation can be abbreviated by the use of exponents. Thus, the set $[5, 5, 1, 1, 1, 1]$ may be written $[5^2, 1^5]$.

$C_r[a, b, \cdots]$ is the number of different combinations of the objects of the set taken r at a time. For example, the problem proposed in Section 1 in notation is to evaluate $C_5[4, 3, 2, 1]$.

3. The ϕ-function for a Set. Associated with each set is a function ϕ which is a polynomial in x, and which serves as a generating function.

$$\phi[a] = 1 + x + x^2 + \cdots + x^a. \tag{2}$$

$$\phi[a, b, c \cdots] = \phi[a] \cdot \phi[b] \cdot \phi[c] \cdots . \tag{3}$$

The degree of ϕ is always n. In fact:

$$\begin{aligned} \phi[a, b, c] &= (1 + x + \cdots + x^a)(1 + \cdots + x^b)(1 + \cdots + x^c), \\ &= A_0 + A_1 \cdot x + A_2 \cdot x^2 + \cdots + A_n \cdot x^n. \end{aligned} \tag{4}$$

The value of any coefficient, A_r, can be found by multiplying out the factors of $\phi[a, b, c]$, and collecting terms of like powers. Before collecting the terms, their number will be $(1+a)(1+b)(1+c)$, and each coefficient is 1. Each term in ϕ is formed by selecting one term in each of the factors.

For example, suppose the term $x^{a'}$ is selected from the first factor, $x^{b'}$ from the second, and $x^{c'}$ from the third. Their product is x to the power $a'+b'+c'$. Now if we are trying to evaluate A_r, we shall select a', b', and c' so that their sum will be r. Also a' is at most a, b' is at most b, and c' is at most c. The value of A_r will be precisely the number of ways of choosing a set of values for a', b', and c'.

In trying to evaluate $C_r[a, b, c]$, one combination will result from choosing a' things from the first subset of a identical things, b' from the second, and c' from the third. The number of possible choices for a', b', and c' is the same as when we were forming terms of $\phi[a, b, c]$. Therefore,

$$C_r[a, b, c] = A_r,$$

the typical coefficient in Equation (4). Moreover, the reasoning may be extended from the set $[a, b, c]$ with only 3 subsets to the general set $[a, b, c, \cdots]$. Therefore,

$$\phi[a, b, c, \cdots] = \sum_{r=0}^{r=n} C_r[a, b, c, \cdots .]\cdot x^r. \tag{5}$$

As a check, apply Equation (5) in the case of the set $[1^n]$, having n distinct elements. $C_r[1^n]$ is the familiar $C_{n,r}$, and

$$\phi[1^n] = (\phi[1])^n = (1 + x)^n. \tag{6}$$

Equation (6), is the familiar relation between the values of $C_{n,r}$ and the binomial coefficients.

Another familiar case is obtained by setting $x=1$ in Equation (5). The sum of the values of $C_r[a, b, c, \cdots]$ for all values of r is $\phi(1)=(1+a)(1+b)(1+c)\cdots$.

4. A Recursion Formula. Next a recursion formula for the determination of $C_r[a, b, c, \cdots .]$ will be derived.

$$\phi[a, \cdots, b, c] = \phi[c]\cdot\phi[a, \cdots, b],$$

$$= (1 + x + x^2 + \cdots + x^c) \sum_{j=0}^{j=n-c} C_j[a, \cdots, b]\cdot x^j.$$

This is an identity, so the coefficients of x^r may be equated.

$$C_r[a, \cdots, b, c] = \sum_{j=r-c}^{j=r} C_j[a, \cdots, b], \tag{7}$$

subject to the following conventions:

1. $[\ ]$ is the empty set, and $n=0$.
2. $\phi[\ ]=1$.
3. $C_0(S)=1$, for any set S.
4. $C_r(S)=0$, if $r<0$, or if $r>n$.

EXAMPLE 1. Find $C_5[4, 3, 2, 1]$.

Solution. The computation is arranged in the following table:

r	$C_r[\ ]$	$C_r[4]$	$C_r[4, 3]$	$C_r[4, 3, 2]$	$C_r[4, 3, 2, 1]$
0	1	1	1	1	1
1	0	1	2	3	4
2	0	1	3	6	9
3	0	1	4	9	15
4	0	1	4	11	20
5	0	0	3	11	22

The value of $C_0[\ ]$ is 1, and $C_r[\ ]=0$ when $r>0$. So the values of $C_r[\ ]$ are quickly filled out. The values in each following column are found by applying Equation (7). The numbers in each column would be the coefficients of ϕ for the same set up to the term in x^5. That is, $\phi[4, 3]=1+2x+3x^2+4x^3+4x^4+3x^5+\cdots$.

EXAMPLE 2. The number of different Bridge hands is $C_{52,13}=635{,}013{,}559{,}600$. From a practical standpoint, a hand is not essentially changed if one card is replaced by another in the same suit, provided both cards are below the ten-spot. Under this practical point of view, how many essentially different Bridge hands are possible?

Solution. The problem, stated in our notation, is to evaluate $C_{13}[8^4, 1^{20}]$. Following is the solution in tabular form for $C_r[8^4]$ for values of r up to 13. The last column of the table is explained later.

r	$C_r[\ ]$	$C_r[8]$	$C_r[8, 8]$	$C_r[8^3]$	$C_r[8^4]$	$C_{13-r}[1^{20}]$
0	1	1	1	1	1	77,520
1	0	1	2	3	4	125,970
2	0	1	3	6	10	167,960
3	0	1	4	10	20	184,756
4	0	1	5	15	35	167,069
5	0	1	6	21	56	125,970
6	0	1	7	28	84	77,520
7	0	1	8	36	120	38,760
8	0	1	9	45	165	15,504
9	0	0	8	52	216	4,845
10	0	0	7	57	270	1,140
11	0	0	6	60	324	190
12	0	0	5	61	375	20
13	0	0	4	60	420	1

It would be out of the question to extend the table for 20 more columns, to bring in the 20 remaining subsets of 1 element each. So now we use Equation (5), and form $\phi[8^4, 1^{20}]$. This is now easy, since the coefficients of one factor, $\phi[8^4]$, are known, and the coefficients of the other factor, $\phi[1^{20}]$, are the easily found binomial coefficients, which we now enter in the last column. Since only the coefficient of x^{13} in $\phi[8^4, 1^{20}]$ is wanted, this is found as the sum:

$$C_{13}[8^4, 1^{20}] = \sum_{r=0}^{r=13} C_r[8^4] \cdot C_{11-r}[1^{20}] = 34{,}033{,}880.$$

Each term of this sum has a special meaning. Take for example the term in which $r=8$. Then $C_8[8^4]=165$, and this is the number of different ways of choosing 8 non-honor cards. $C_{13-8}[1^{20}]=15{,}504$ is the number of different ways of choosing the 5 honor cards. Their product, (165)(15,504), is the number of essentially different hands containing 8 non-honors and 5 honors.

A NOTE ON LEAST SQUARES

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In the usual treatment of the method of least squares it is recommended that the so-called normal equations be computed explicitly using, for instance, Crelle's multiplication table and a table of squares, and that these equations then be solved. In this age of computing machinery a slightly different form of the result which gives the solution of the normal equations directly in terms of the coefficients of the given equations of condition may be helpful. The method is easily remembered, is similar to Cramer's Rule in form, and may be thought of as a generalization of it in that when the number of equations of condition is equal to the number of unknowns the method reduces to Cramer's Rule. The treatment in this form may be of interest to teachers to furnish exercises as it involves the multiplication of matrices, the evaluation of determinants, Cramer's Rule, and least squares theory.*

The equations of condition are a set of linear equations whose number is greater than or equal to the number of unknowns, and the equations are not strictly compatible with each other. The solution desired is a set of values of the unknowns which satisfies the equations as nearly as possible; that is, in the least squares sense. The equations of condition are:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, 3, \cdots, m),\ m \geqq n, \tag{1}$$

and may be written as:

$$AX = B, \tag{2}$$

where A is the m by n matrix of the coefficients; X, the one by n column matrix of the unknowns; and B the one by m column matrix of the right-hand constants.

Transpose the b's of (1) to the left sides leaving the right members zero. Since for independent equations with $m>n$ no solution that satisfies all the equations may be found, any approximate solution—in this case, the least squares solution—will not make the right members zero but small. Call these small quantities ϵ_i.

* For similar discussions, see: Holzinger and Harmon, Factor Analysis, University of Chicago Press, 1941, p. 290 ff.; and Arley and Buch, Probability and Statistics, John Wiley and Sons, 1950, p. 185 ff.

$$\sum_{j=1}^{n} a_{ij}x_j - b_i = \epsilon_i, \qquad i = 1, 2, 3, \cdots, m. \tag{3}$$

Let

$$\epsilon = \sum_{i=1}^{m} \epsilon_i^2 \tag{4}$$

then, for the least squares solution, ϵ is a minimum, and

$$\frac{\partial \epsilon}{\partial x_j} = 0, \qquad j = 1, 2, 3, \cdots, n. \tag{5}$$

This constitutes a set of n linear equations in n unknowns (the normal equations):

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ik}a_{ij}\right)x_j = \sum_{i=1}^{m} a_{ik}b_i, \qquad k = 1, 2, \cdots, n. \tag{6}$$

These may be written in matrix notation:

$$A^TAX = A^TB. \tag{7}$$

If A^TA is non-singular the solution is

$$X = (A^TA)^{-1}A^TB. \tag{8}$$

By writing M_j for the matrix A with the jth column replaced by the column matrix B, we may write (8) in the form:

$$x_j = \frac{|A^TM_j|}{|A^TA|} \qquad j = 1, 2, \cdots, n \tag{9}$$

Note that $|A^TA|$ is always symmetric.

When $m = n$ the matrices are square and since the determinant of a product of *square* matrices is equal to the product of the determinants of the matrices,

$$x_j = \frac{|A^T||M_j|}{|A^T||A|} = \frac{|M_j|}{|A|} \tag{10}$$

which is Cramer's Rule.

The following examples illustrate the use of (9) above.

EXAMPLE 1. (Taken from *The Calculus of Observations*, Whittaker and Robinson, The Method of Least Squares, p. 212.)

$$
\begin{aligned}
4.91x - 59.0\ y &= -339.8 \\
2.72x - 2.70y &= -47.5 \\
0.05x + 32.4y &= 262.5 \\
-2.91x + 27.7\ y &= 152.9 \\
-4.77x + 1.50y &= -27.9
\end{aligned}
$$

$$
x = \frac{\begin{vmatrix} \begin{bmatrix} 4.91 & 2.72 & 0.05 & -2.91 & -4.77 \\ -59.00 & -2.70 & 32.40 & 27.70 & 1.40 \end{bmatrix} & \begin{bmatrix} -339.8 & -59.0 \\ -47.5 & -2.70 \\ 262.5 & 32.4 \\ 152.9 & 27.7 \\ -27.9 & 1.40 \end{bmatrix} \end{vmatrix}}{\begin{vmatrix} \begin{bmatrix} 4.91 & 2.72 & 0.05 & -2.91 & -4.77 \\ -59.00 & -2.70 & 32.40 & 27.70 & 1.40 \end{bmatrix} & \begin{bmatrix} 4.91 & -59.0 \\ 2.72 & -2.70 \\ 0.05 & 32.4 \\ -2.91 & 27.7 \\ -4.77 & 1.40 \end{bmatrix} \end{vmatrix}}
$$

$$
x = \frac{\begin{vmatrix} -2096.349 & -382.699 \\ 32877.72 & 5307.3 \end{vmatrix}}{\begin{vmatrix} 62.73 & -382.699 \\ -382.699 & 5307.3 \end{vmatrix}} = \frac{1456317.5}{186468.404} = 7.81
$$

Similarly, $y = 6.76$

EXAMPLE 2. (Taken from *Numerical Mathematical Analysis*, J. B. Scarborough, p. 366.)

Find a formula of the form

$$y = a + bx + cx^2$$

which will fit the following data:

x.	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
y	3.1950	3.2299	3.2532	3.2611	3.2516	3.2282	3.1807	3.1266	3.0594	2.9756

Substituting these values of x and y into the quadratic equation $y = a + bx + cx^2$ gives ten linear equations of condition in a, b, and c whose solution is obtained by the above process: $a = 3.1951$; $b = .44253$; $c = -.76530$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 936. *Proposed by R. E. Horton, Los Angeles City College*

A wheel of radius a with a flange of radius b is rolling without slipping along a straight rail. On the corner of the rail a piece projects which scratches a mark on the face of the flange. Identify the curve of the mark so formed. What conditions on a and b are necessary in order that the mark on the flange will be a simple closed curve?

E 937. *Proposed by I. N. Herstein, University of Kansas*

If p is prime and $n \geqq p$, then

$$n! \sum_{pi+j=n} 1/p^i i!j! \equiv 0, \text{ mod } p.$$

E 938. *Proposed by H. F. Sandham, Trinity College, Ireland*

If $x = \sum_{i=1}^{n} 1/i$, prove that $n = [e^{x-\gamma}]$, where γ is Euler's constant, and $[a]$ denotes the integral part of a.

E 939. *Proposed by P. J. Schillo, University of Buffalo*

Circles C_0, C_1, C_2, $\cdots$, having radii r_0, r_1, r_2, $\cdots$, are tangent to a circle C, of radius $2r$, and to a diameter of C. Circle C_0 passes through the center of C, and C_n is tangent externally to C_{n-1}. Show that

$$r_n = 4w^n r/(w^n + 1)^2,$$

where $w = 3+2\sqrt{2}$, and that r/r_n is an integer.

E 940. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Being given a tetrahedron $ABCD$, an arbitrary point P, and four coplanar points A', B', C', D' lying in the planes of the faces BCD, CDA, DAB, ABC, show that the lines PA', PB', PC', PD' intersect the spheres $PBCD$, $PCDA$, $PDAB$, $PABC$ in four points cospherical with P.

SOLUTIONS

A Polygonal Path of Arbitrarily Small Length

E 906 [1950, 184]. *Proposed by Vern Hoggatt, Oregon State College*

Let S be a bounded infinite point set. Show that there exists a polygonal

path of arbitrarily small length passing through an infinite number of points of S.

Solution by Norman Miller, Queen's University, Ontario. The Bolzano-Weierstrass theorem guarantees the existence of a limit point P of S. With P as center draw a circle of diameter $\epsilon/2$, and within it select a point of S. Within a second circle with center P and diameter $\epsilon/4$ select another point of S. Continue this process with circles of diameters $\epsilon/8$, $\epsilon/16$, $\cdots$. The polygonal path formed by consecutively joining the infinite set of selected points has length $l<\epsilon/2+\epsilon/4+\cdots=\epsilon$.

Also solved by F. Bagemihl, David Ellis, J. B. Feidner, M. D. Montgomery, C. S. Ogilvy, C. F. Pinzka, L. A. Ringenberg, R. M. Simons, O. D. Smith, Charles Solky, G. H. M. Thomas, S. S. Walters, and the proposer.

As pointed out by Ellis, the above solution can be generalized to solve the problem where S is any bounded infinite subset of a finitely compact metric space (metric space in which bounded subsets are compact). It is to be noted that the proof assumes Zermelo's axiom of choice.

Sum of Two Squares in Three Different Ways

E 909 [1950, 184]. *Proposed by W. E. Buker, Perry High School, Pittsburgh, Pa.*

Find positive integers satisfying

$$N = p^2 + k_1^2 = (p+1)^2 + k_2^2 = (p+2)^2 + k_3^2.$$

I. *Solution by P. Somanadham, Vizianagram, India.* Clearly $k_1>k_2>k_3$ and k_1 and k_3 are of the same parity. Setting $a=(k_1+k_3)/2$ and $b=(k_1-k_3)/2$ we have $k_1=a+b$ and $k_3=a-b$, whence

$$p^2 + (a+b)^2 = (p+2)^2 + (a-b)^2$$

or

$$p = ab - 1.$$

Since

$$p^2 + (a+b)^2 = (p+1)^2 + k_2^2$$

we have

$$(ab-1)^2 + (a+b)^2 = (ab)^2 + k_2^2$$

or

$$k_2^2 = a^2 + b^2 + 1.$$

The right side will be a perfect square if $a=2t^2$ and $b=2t$. We then have the parametric solution

$$k_2 = 2t^2 + 1, \qquad p = 4t^3 - 1, \qquad k_1 = 2t(t+1), \qquad k_3 = 2t(t-1),$$

$$N = 16t^6 + 4t^4 + 4t^2 + 1, \qquad t > 1.$$

This does not give all solutions, however, since the particular solution

$$(539)^2 + (48)^2 = (540)^2 + (35)^2 = (541)^2 + (12)^2$$

cannot arise from an integral value of t.

II. *Solution by Elijah Swift, University of Vermont.* We easily find that

$$k_1^2 + k_3^2 = 2(k_2^2 - 1). \tag{1}$$

Noting that the square of an odd number is of the form $8a+1$ we see that k_2 must be odd and k_1 and k_3 both even. Consequently, any solution of (1) will determine p as an integer and yield a solution of the given equations. For $2(k_2^2-1)$ to be the sum of two squares it must be of the form 2^rMQ^2, where $r \geqq 0$, the factors of M are primes of the form $4x+1$, and the factors of Q are primes of the form $4x+3$. Substituting odd numbers for k_2 we find eight values of $k_2<100$ such that $2(k_2^2-1)$ satisfies the above conditions. These give the following eleven solutions.

k_1	k_2	k_3	p	N
12	9	4	31	1105
24	19	12	107	12025
40	33	24	255	66625
48	35	12	539	292825
60	51	40	499	252601
68	51	24	1011	1026745
72	51	4	1291	1671865
84	73	60	863	751825
104	81	48	2127	4534945
112	81	24	2991	8738625
112	99	84	1371	1892185

Also solved by D. H. Browne, R. L. Caskey, Monte Dernham, W. A. Ferguson, Daniel Finkel, G. B. Huff, Free Jamison, Sam Kravitz, Roger Lessard, Leonard Morgenstern, Prasert Na Nagara, C. S. Ogilvy, K. S. Rao, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, R. M. Simons, O. D. Smith, G. W. Walker, and the proposer. Most of these gave the parametric solution in I above. A few provided more or less tentative processes for finding all solutions. Walker listed all solutions (twenty-three of them) for $N<10^8$. Jamison found some two-parameter solutions.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4412. *Proposed by Jacques Dutka, Rutgers University*

Of the freshmen admitted to a certain college last fall it was found that:
74% were seventeen years of age or older,
89% weighed 125 pounds or more,
78% measured $5\frac{1}{2}$ feet tall or more.

Let p be the probability that a freshman chosen at random had all three characteristics. Determine bounds for the value of p, and generalize the problem.

4413. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $a_1=2\cdot 3$, $a_2=3\cdot 5$, $a_3=5\cdot 7$, $\cdots$, $a_k=p_k\cdot p_{k+1}$, $\cdots$, where p_k is the kth prime. Denote by $f(n)$ the number of integers $\leqq n$ composed entirely of the a's (i.e., the integers of the form $\prod a_i^{\alpha_i}$, $0\leqq\alpha_i$). Prove that

$$f(n) = cn^{1/2} + o(n^{1/2}),$$

where $\frac{1}{2}<c<1$.

4414. *Proposed by Lucien Droussent, Clermont Ferrand, France*

The trilinear polar, with respect to triangle T, of any point P on the circumscribed Steiner ellipse of T is tangent to the conjugate conic of T that has P for center. (The circumscribed Steiner ellipse of a triangle ABC is the circumscribed ellipse whose tangents at A, B, C are parallel to the opposite sides BC, CA, AB.)

SOLUTIONS

Root of Maximum Modulus

4331 [1949, 111]. *Proposed by V. F. Ivanoff, San Francisco*

Let the equation, $x^n+p_1x^{n-1}+\cdots+p_n=0$, possess a root x_1 whose modulus exceeds that of every other root of the equation. Prove that

$$\lim_{k\to\infty}\frac{\Delta_k}{\Delta_{k-1}} = -x_1,$$

where Δ_k is the k-rowed determinant ($k>n$)

$$\begin{vmatrix} p_1 & p_2 & \cdot & \cdot & p_n & 0 & \cdot & \cdot & 0 & 0 \\ 1 & p_1 & p_2 & \cdot & \cdot & p_n & \cdot & \cdot & 0 & 0 \\ 0 & 1 & p_1 & p_2 & \cdot & \cdot & p_n & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & p_1 & \cdot & \cdot & \cdot & p_{n-1} & np_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 & p_1 & 2p_2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & p_1 \end{vmatrix}.$$

I. *Solution by Roger Lessard.* École Polytechnique, Montreal, Canada. Developing Δ_k by the first column we get two determinants; keeping the first and developing the second by its first column, and so on, we get finally

$$\Delta_k = \sum_{j=1}^{n} (-1)^{j-1} p_j \Delta_{k-j},$$

a recursion formula expressing Δ_k as a function of the n preceding determinants. (If $k \leqq 2n$, certain Δ_h appear which are not defined in the proposal. A consistent definition, however, is obvious.) We may write the above equation as

$$\sum_{j=0}^{n} (-1)^j p_j \Delta_{k-j} = 0, \qquad p_0 = 1. \tag{1}$$

But this is a classical problem, studied by Daniel Bernoulli (*Commentarii Acad. Sc. Petropol*, III, 1732). The ratio of two successive terms of (1) tends to a limit which is that root X_1 (if one exists) whose modulus exceeds that of every other root of the characteristic equation of (1), namely

$$\sum_{j=0}^{n} (-1)^j p_j X^{n-j} = 0. \tag{2}$$

Since this equation has for its roots the negatives of those of the given equation we have, as desired.

$$\lim_{k \to \infty} \frac{\Delta_k}{\Delta_{k-1}} = X_1 = -x_1.$$

II. *Solution by Robert Steinberg, University of California at Los Angeles.* Let

$$\frac{p_1 + 2p_2 x + \cdots + np_n x^{n-1}}{1 + p_1 x + p_2 x^2 + \cdots + p_n x^n} = \sum_{k=0}^{\infty} a_k x^k.$$

Cross-multiplication and the equating of the coefficients of like powers of x gives a system of linear equations whose solution is $a_{k-1} = (-1)^{k-1} \Delta_k$. On the other hand, if we let

$$f(x) = 1 + p_1x + p_2x^2 + \cdots + p_nx^n = \prod_{i=1}^{n} (1 - x_ix),$$

then

$$\frac{f'(x)}{f(x)} = -\sum_{i=1}^{n} \frac{x_i}{1 - x_ix} = -\sum_{i=1}^{n}\sum_{j=0}^{\infty} x_i^{j+1}x^j = -\sum_{j=0}^{\infty} x^j \sum_{i=1}^{n} x_i^{j+1}.$$

Thus $a_{k-1} = -\sum_1^n x_i^k$. Hence $\Delta_k = (-1)^k \sum_1^n x_i^k$ and

$$\lim \frac{\Delta_k}{\Delta_{k-1}} = -\lim \frac{\sum_{i=1}^{n} x_i^k}{\sum_{i=1}^{n} x_i^{k-1}} = -x_1.$$

Also solved by the Proposer.

Editorial Note. The Proposer's solution proceeds by first showing that $\Delta_k = (-1)^k S_k$, where S_k is the sum of the kth powers of all the roots. C. R. Phelps points out the intimate connection with Graeffe's method of approximating roots. A. W. Goodman refers to similar problems in F. Riesz, *Les Systemes d'Equations a une Infinite d'Inconnues*, Paris 1913, pp. 9–11. D. H. Browne compares the present result with an earlier editorial note, [1942, 463].

Sequences of pth Powers

4333 [1949, 112]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In every system of numeration whose base B has the form am^p+1, each of the p-digit numbers

$$(am^{p-1} + 1)^p,\ (2am^{p-1} + 1)^p,\ (3am^{p-1} + 1)^p,\ \cdots,\ [(m - 1)am^{p-1} + 1]^p,$$

gives rise to an infinite sequence of pth powers. The law of formation is very simple and is sufficiently clear from the following example:

If $B = 97 = 6\cdot4^2+1 = 12\cdot2^3+1 = 6\cdot2^4+1 = 3\cdot2^5+1$, then we have

$\overline{25}^2 = 6\ \overline{43},\quad \overline{24}\ \overline{25}^2 = 6\ 6\ \overline{42}\ \overline{43},\quad \overline{24}\ \overline{24}\ \overline{25}^2 = 6\ 6\ 6\ \overline{42}\ \overline{42}\ \overline{43}, \cdots$

$\overline{49}^2 = \overline{24}\ \overline{73},\quad \overline{48}\ \overline{49}^2 = \overline{24}\ \overline{24}\ \overline{72}\ \overline{73},$

$\overline{48}\ \overline{48}\ \overline{49}^2 = \overline{24}\ \overline{24}\ \overline{24}\ \overline{72}\ \overline{72}\ \overline{73}, \cdots$

$\overline{73}^2 = \overline{54}\ \overline{91},\quad \overline{72}\ \overline{73}^2 = \overline{54}\ \overline{54}\ \overline{90}\ \overline{91},$

$\overline{72}\ \overline{72}\ \overline{73}^2 = \overline{54}\ \overline{54}\ \overline{54}\ \overline{90}\ \overline{90}\ \overline{91}, \cdots$

$\overline{49}^3 = \overline{12}\ \overline{48}\ \overline{85},\quad \overline{48}\ \overline{49}^3 = \overline{12}\ \overline{12}\ \overline{48}\ \overline{48}\ \overline{84}\ \overline{85},$

$\overline{48}\ \overline{48}\ \overline{49}^3 = \overline{12}\ \overline{12}\ \overline{12}\ \overline{48}\ \overline{48}\ \overline{48}\ \overline{84}\ \overline{84}\ \overline{85}, \cdots$

$\overline{49}^4 = 6\ \overline{30}\ \overline{66}\ \overline{91},\quad \overline{48}\ \overline{49}^4 = 6\ 6\ \overline{30}\ \overline{30}\ \overline{66}\ \overline{66}\ \overline{90}\ \overline{91},$

$\overline{48}\ \overline{48}\ \overline{49}^4 = 6\ 6\ 6\ \overline{30}\ \overline{30}\ \overline{30}\ \overline{66}\ \overline{66}\ \overline{66}\ \overline{90}\ \overline{90}\ \overline{91}, \cdots$

$\overline{49}^5 = 3\ \overline{18}\ \overline{48}\ \overline{78}\ \overline{94},\quad \overline{48}\ \overline{49}^5 = 3\ 3\ \overline{18}\ \overline{18}\ \overline{48}\ \overline{48}\ \overline{78}\ \overline{78}\ \overline{93}\ \overline{94},$

$\overline{48}\ \overline{48}\ \overline{49}^5 = 3\ 3\ 3\ \overline{18}\ \overline{18}\ \overline{18}\ \overline{48}\ \overline{48}\ \overline{48}\ \overline{78}\ \overline{78}\ \overline{78}\ \overline{93}\ \overline{93}\ \overline{94}, \cdots.$

Establish the general rule.

Solution by E. P. Starke, Rutgers University. Put

$$n_0 = kam^{p-1} + 1, \qquad k < m,$$

$$n_s = 1 + (n_0 - 1)(B^s + B^{s-1} + \cdots + B + 1) = \frac{n_0 - 1}{B - 1}(B^{s+1} - 1) + 1.$$

Then we have

$$a = \frac{B-1}{m^p}, \qquad \frac{n_0 - 1}{k} = \frac{B-1}{m}, \qquad \frac{n_s - 1}{k} = \frac{B^{s+1} - 1}{m}.$$

The following formulas are easily computed:

$$n_0^p = \left[1 + \frac{k}{m}(B-1)\right]^p = 1 + a\sum_{j=0}^{p-1} m^j \binom{p}{j} k^{p-j}(B-1)^{p-j-1},$$

$$n_s^p = \left[1 + \frac{k}{m}(B^{s+1} - 1)\right]^p$$

$$= 1 + a(B^s + B^{s-1} + \cdots + B + 1)\sum_{j=0}^{p-1} m^j \binom{p}{j} k^{p-j}(B^{s+1} - 1)^{p-j-1}.$$

Now suppose n_0^p, written in the scale of B, is $d_1 d_2 d_3 \cdots d_p$. Then $n_s^p - 1$ can be obtained from $n_0^p - 1$ by replacing each B by B^{s+1} and multiplying the result by $B^s + B^{s-1} + \cdots + B + 1$. The result is clearly a number $d_1 d_1 \cdots d_1 d_2 \cdots d_2 d_3 \cdots d_3 \cdots d_p \cdots d_p$ with each digit now written $s+1$ times.

Trigonometric Sum and Product

4335 [1949, 112]. *Proposed by D. D. Wall, Cambridge, Massachusetts*

If n is a prime greater than 3, show that

$$\prod_{x=1}^{n}\left(1 + 2\cos\frac{2\pi x}{n}\right) = 3, \tag{1}$$

$$\sum_{x=1}^{n} \sec\frac{2\pi x}{n} = (-1)^{(n-1)/2} n. \tag{2}$$

The restrictions on n can be lessened somewhat in each case.

Solution by N. J. Fine, University of Pennsylvania. Let n be a positive integer. If n is a multiple of 3, the product (1) is zero since it contains the factor

$$1 + 2\cos 2\pi\frac{(n/3)}{n} = 0.$$

Otherwise, set $w = e^{2\pi i/n}$. Then

$$\prod_{k=1}^{n}\left(1+2\cos\frac{2\pi k}{n}\right)=\prod_{k=1}^{n}(1+w^k+w^{-k})=w^{-n(n+1)/2}\prod_{k=1}^{n}(1+w^k+w^{2k})$$

$$=(-1)^{n+1}3\,\frac{\prod_{k=1}^{n-1}(1-w^{3k})}{\prod_{k=1}^{n-1}(1-w^k)}=(-1)^{n+1}3,$$

since $3k$ runs over all residues (mod n) except 0 as k does.

For the second part, let n be odd and $w=e^{2\pi i/n}$. Then

$$\sum_{k=1}^{n}\sec\frac{2\pi k}{n}=2\sum_{k=1}^{n}\frac{1}{w^k+w^{-k}}=2\sum_{k=1}^{n}\frac{w^k}{1+w^{2k}}$$

$$=\sum_{k=1}^{n}w^k\left(\frac{1+w^{2kn}}{1+w^{2k}}\right)=\sum_{k=1}^{n}w^k\sum_{r=0}^{n-1}(-1)^r w^{2kr}$$

$$=\sum_{r=0}^{n-1}(-1)^r\sum_{k=1}^{n}(w^{2r+1})^k=(-1)^{(n-1)/2}n,$$

since the inner sum is 0 if n is not a divisor of $2r+1$, and is n if n is a divisor of $2r+1$, and the latter is true for $0\leqq r\leqq n-1$ only when $r=(n-1)/2$.

Also solved by N. C. Scholomiti and the Proposer.

Motion Through a Planet of Constant Density

4342 [1949, 268]. *Proposed by R. J. Walker, Cornell University*

A spherical planet whose density at any point P is a function only of the distance of P from the center of the planet has the following property. If a straight frictionless tunnel is bored between two points on the planet's surface the time required for an object to slide from one of these points to the other is independent of the positions of the points. Prove that the planet has constant density.

Solution by M. Kac, H. Pollard, and the proposer, Cornell University. Let a be the radius of the sphere, $\rho(r)$ the density at a point P, r units from the center, and $f(r)$ the acceleration due to gravity at P. Then

$$f(r)=\frac{k}{r^2}\int_0^r 4\pi\rho(s)s^2ds. \tag{1}$$

If a unit mass falls along any frictionless path from the surface to P its velocity at P is given by

$$\frac{1}{2}[V(r)]^2=\int_r^a f(s)ds. \tag{2}$$

Consider a straight frictionless tunnel whose distance from the center is x. If y is the distance from P to the center of the tunnel we have

$$y^2 = r^2 - x^2,$$

$$V(r) = \frac{dy}{dt} = \frac{r}{\sqrt{r^2 - x^2}} \frac{dr}{dt}.$$

Hence if T is the time required to slide through the tunnel,

$$\int_x^a \frac{r\,dr}{V(r)\sqrt{r^2 - x^2}} = \frac{T}{2}. \tag{3}$$

To solve this for $V(r)$ we use a variation of the Abel inversion formula. (See Bôcher, *An Introduction to the Study of Integral Equations*.) Multiply (3) by $x/\sqrt{x^2-y^2}$ and integrate with respect to x from y to a, y being an arbitrary number between 0 and a.

$$\int_y^a dx \int_x^a \frac{x r\,dr}{V(r)\sqrt{x^2 - y^2}\sqrt{r^2 - x^2}} = \int_y^a \frac{T}{2} \frac{x\,dx}{\sqrt{x^2 - y^2}}$$

$$= \frac{T}{2}\sqrt{a^2 - y^2}.$$

Interchanging the order of integration gives

$$\frac{T}{2}\sqrt{a^2 - y^2} = \int_y^a \frac{r\,dr}{V(r)} \int_y^r \frac{x\,dx}{\sqrt{x^2 - y^2}\sqrt{r^2 - x^2}}$$

$$= \frac{\pi}{2} \int_y^a \frac{r\,dr}{V(r)}.$$

On differentiating with respect to y we get

$$V(y) = \frac{\pi}{T}\sqrt{a^2 - y^2}.$$

Differentiating (2) now gives

$$f(r) = \frac{\pi^2}{T^2} r,$$

and putting this in (1) and differentiating gives

$$\rho(r) = \frac{3\pi}{4kT^2}.$$

Also solved by R. N. Haskell, and Fritz Herzog.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Sampling Methods for Censuses and Surveys. By Frank Yates. London, Charles Griffin & Co. Ltd., 1949. 14+318 pages. 24 s.

"This book has been written primarily for those who have little or no previous training in mathematical statistics, but who have some training or experience in the presentation and handling of statistical data. It is consequently not written in the form of a mathematical treatise, and mathematical proofs have not been included. On the other hand an attempt has been made to cover all the modern developments of sampling theory which are of importance in census and survey work, and to give an adequate discussion of the complexities that are encountered in their practical application." This statement, with which the author begins his preface, accurately describes the book.

The character of the work is influenced by its origin in a request of the United Nations Sub-Commission on Statistical Sampling that a manual be prepared for use in the projected 1950 world censuses of agriculture and population. It is concerned primarily with the techniques of planning, execution, and analysis of censuses and surveys. The first half of the book is devoted to a description of the different methods of sampling and to various non-mathematical procedural matters. The second half consists of recipes for estimating population values, sampling errors, and efficiencies. Numerous examples (almost entirely agricultural) are worked out with clarity and completeness, often being carried along from one section of the book to another. There is a classified bibliography on sampling, which is based on a more complete one prepared by the Statistical Office of the United Nations.

It is a regrettable fact that many workers in statistics "have little or no previous training in mathematical statistics." The large and growing demand for statistical work implies that the situation is not likely to mend automatically. Of course the only satisfactory long run solution is to insist that statistical workers secure the necessary training in mathematics. A "statistician" without this basic equipment is as much a public hazard as the quack doctor. But it will certainly be some time before professional standards can be fully enforced in the field of statistics, and meanwhile there is a real need to increase the effectiveness of the mathematically untrained statistician. One solution is the statistical "cook book." Here the formulas and techniques are presented categorically with exact instructions as to what to do, but practically without justification. The reader with "little or no previous training" is still in this category after mastering such a book, although he is acquainted with new terminology and "tricks of the trade." Moreover, he is likely to be confirmed in his impression

that "mathematical statistics" is a mystical religion understandable only to its high priests. Another and, it seems to this writer, a better solution is to try by means of precise statement, heuristic arguments, and intuitively appealing examples to increase the reader's appreciation of the meaning and justification of statistical procedures. He may thus gain a "feel" for the theory which can be a helpful prelude or complement to the rigorous mathematics essential to a thorough understanding. Since the book under review is largely in the "cook book" category, the writer would be more inclined to recommend it as an up-to-date compilation of techniques for the mathematically trained statistician than as a text for those who stand in need of basic training in mathematical statistics.

KENNETH MAY

Higher Algebra for the Undergraduate. By Marie J. Weiss. New York, John Wiley, 1949. 8+165 pp. $3.75.

Higher Algebra for the Undergraduate is designed for a one year course in algebra for students who have completed a year of calculus. The author's purpose has been to give an introduction to simpler concepts of algebra at a level of difficulty about the same as that of the traditional courses in theory of equations. To keep the presentation elementary the author has placed emphasis on careful exposition with many illustrative examples, on exercises for homework, and on keeping the number of new ideas introduced at any one time to a minimum. The smoothness of the exposition and careful choice of subject matter achieved by the author result at least in part from her testing of preliminary versions for a number of years in her own classes.

The first chapter treats the integers. In line with the elementary character of the text there is no attempt at an axiomatic development of the properties of the natural numbers, but a complete set of postulates is set down. The extension from natural numbers to integers is treated quite fully by the introduction of ordered pairs with a suitable equivalence relation. The frequent use of equivalence relations is an interesting feature of the text. Although the connection between equivalence relations and partitions of sets is never stated as a proposition in set theory, it is sufficiently emphasized in specific cases to give the student an excellent background for a more complete treatment in some later course. The second chapter begins with the extensions from integers to rationals to reals to complex numbers. The first and third of these extensions are made by the use of ordered pairs. The extension from rationals to reals is a mixture of Kronecker, Dedekind, and geometric intuition and is in the reviewer's opinion the weakest section in the entire book. The chapter closes with a discussion of roots of unity.

Groups are introduced in the third chapter. The treatment begins with the group axioms and develops some of the elementary theorems on abstract groups and permutation groups. This first treatment includes the concepts of isomorphism, subgroup, coset, cyclic group, and regular permutation group. The fourth chapter brings in rings, integral domains, and fields making considerable use of concrete examples from the number systems already discussed in the first two

chapters. The definition of "indeterminate" is not too satisfactory logically although it is unlikely to lead any but the unusually gifted student into difficulty. The fifth chapter treats polynomials over a field, with primary emphasis on divisibility theory. There is a brief discussion of the determination of rational roots of an equation with integer coefficients; the elementary symmetric functions are defined (but not symmetric functions in general); and the idea of formal derivative is introduced and applied to get the usual criterion for multiple factors of a polynomial and to get Taylor's Theorem for polynomials.

In chapter six matrices and some matrix algebra are introduced. The definition of rank is approached via the concept of row (column) equivalence and thus makes no use of determinants. The general theory for systems of linear equations is handled in matrix language. Then in chapter seven determinants are introduced as functions of square matrices. The author carefully limits the use of the word "determinant" to this function and never uses "determinant" where "matrix" is meant. The proof of the theorem on the determinant of a product is made to rest on the earlier results about "elementary" matrices. Then the determinant criterion for rank of a matrix and Cramer's Rule are given. The chapter concludes with brief treatments of similarity of matrices and of polynomials with matrix coefficients which include the Cayley-Hamilton Theorem and the diagonalization theorem for matrices without repeated eigenvalues.

The final chapter returns to abstract algebra with primary attention given to the fundamental homomorphism theorems for groups, rings and immediately related concepts and topics. Automorphisms of groups are briefly treated.

The reviewer agrees with the author's position that much of the content of the traditional course in theory of equations is more analysis than algebra. However, he feels that the balance of the text would have been improved if enough of the traditional theory of equations (in particular cubic and quartic equations) had been retained so that the prospective high school teacher could find in a one year course the necessary background material for school algebra as well as an introduction to abstract concepts.

This book by Professor Weiss is a welcome and important addition to the list of advanced undergraduate texts in algebra. To the reviewer it seems to be the simplest introduction to abstract algebra yet available. It is recommended for courses which attempt to bridge the gap between college algebra and modern abstract algebra, especially in situations where some of the other modern algebra texts have been found too advanced.

R. M. THRALL

An Introduction to the Algebra of Vectors and Matrices. By T. L. Wade. Tallahassee, Florida, 1949. 6+97 pages. $2.25.

To quote from the preface: "the mathematical background of the student using this book is assumed to be quite limited, namely, a first course in analytic geometry and acquaintanceship with the usual theorems on determinants."

The book presents a fairly extensive view of vectors and matrices, as the following chapter headings will indicate: IV. Vectors of N Dimensions; VII. Groups, Matrices and Transformations; VIII. The Characteristic Equation of a Matrix; X. Some Applications of Matrix Algebras. There are frequent exercises.

I feel that this textbook offers the beginning student a useful body of information, unencumbered by the (at first bewildering) abstractions of modern algebra. In the hands of a well-informed and skillful teacher, it should prove valuable. At the same time it should be stated that the text would benefit from a thorough revision. The typographical errors are covered by three mimeographed sheets issued by the author, but there are other defects as well. For example, the definition of a field is incorrect as it stands. Again, the definition of the product of two permutations is given two paragraphs later than a discussion of the representation of a permutations as a product of cycles.

R. H. BRUCK

NEW BOOKS RECEIVED

Phenomena, Atoms and Molecules. By I. Langmuir. New York, Philosophical Library, 1950. 12+436 pp. $10.00.

Introduction to the Theory of Probability and Statistics. By N. Arley and K. R. Buch. New York, John Wiley and Sons, 1950. 12+236 pp. $4.00.

Giant Brains or Machines That Think. By E. C. Berkeley. New York, John Wiley and Sons, 1949. 16+270 pp. $4.00.

Plane and Spherical Trigonometry. Fifth Edition. By C. I. Palmer, C. W. Leigh and S. H. Kimball. New York, McGraw-Hill, 1950. 14+102 pp. $3.25.

L'analyse Mathematique. ("Que sais-je? " no. 378). By A. Delachet. Paris, Presses Univerisiaires de France, 1949. 118 pp.

Vorlesungen über höhere Mathematik. Vol. 1. By A. Duschek. Vienna, Springer, 1949. 10+395 pp. $8.70.

Fourier Methods. By P. Franklin. New York, McGraw-Hill, 1949. 10+289 pp. $4.00.

Grundzuge der theoretischen Logik. Third Revised Edition. By D. Hilbert and W. Ackerman. Berlin, Springer, 1949. 8+155 pp. DM 19.80.

Matrix Analysis of Electric Networks (Harvard Monographs in Applied Science, no. 1). By P. Le Corbeiller. Harvard University Press, 1950. 12+112 pp. $3.00.

Determinanten und matrizen. Third Revised Edition. By F. Neiss. Berlin, Springer, 1948. 8+111 pp. DM 6.

Problemi di secondo grado risolti e discussi. By A. Sartori. Turin, Paravia, 1949. 5+163 pp. Lire 600.

Principles of Finance and Investment. By L. G. Whyte. Cambridge University Press, 1950, xi+192 pp. $2.50.

Statistics, Volume I. By N. L. Johnson and H. Tetley. Cambridge University Press, 1950, xii+294 pp. $4.00.

Out of My Later Years. By Albert Einstein. New York, Philosophical Library, 1950. 8+282 pp. $4.75.

Elements of Analytic Geometry. Third Edition. By C. E. Love. New York, MacMillan, 1950. 8+218 pp. $2.75.

Plane and Spherical Trigonometry. Revised Edition. By J. A. Northcott. New York, Rinehart, 1950. 10+94 pp. $3.50.

Transcendental Numbers. By C. L. Siegel. Princeton University Press, 1949. 8+102 pp. $2.00.

Theory of Sets. Translated by F. Bagemihl (from the Second German Edition by E. Kamke). Dover Publications, 1950. x+150 pp. $2.45.

Numerical Solutions of Differential Equations. By H. Levy and E. A. Baggott. Dover Publications, 1950. viii+238 pp. $3.00.

Apercu Historique de la Trigonometrie Rectiligne et Spherique. By A. Gloden. Luxembourg, Worre-Mertens, 1949. 1+18 pp.

Apercu Historique des Multigrades. By A. Gloden. Luxembourg, Worre-Mertens, 1949. 18+23 pp.

Contributions to the Theory of Nonlinear Oscillations. By S. Lefschetz. (Annals of Mathematics Studies, no. 20). Princeton University Press, 1950. 10+350 pp. $4.00.

L'analysis situs et la Geómetrie Algébrique. By S. Lefschetz. Paris, Gauthier-Villars, 1950. 6+154 pp.

Albert Einstein: Philosopher-scientist. (Library of Living Philosophers, vol. 7). Edited by P. A. Schilpp. Evanston, Library of Living Philosophers, 1949. 16+781 pp. $8.50.

Teorijska fizika i struktura materije. By I. Supek. Zagreb, Nakladni Zavod Hrvatske, 1949. 688 pp.

Elements of Calculus. T. S. Peterson. New York, Harper, 1950. 10+369 pp. $3.50.

Tables of the Bessel Functions of the First Kind of Orders Sixty-four through Seventy-eight. The Annals of the Computation Laboratory of Harvard University, volume XIII. Cambridge, Harvard University Press, 1949, 2+566 pp. $8.00.

Science and the Goals of Man. By A. Rapoport. New York, Harper, 1950. 29+262 pp. $3.50.

Electromagnetic Theory. (Proceedings of symposia in applied mathematics, volume 2). New York, American Mathematical Society, 1950. 2+91 pp. $3.00.

Algebraic Curves. By R. J. Walker. Princeton University Press, 1950. 11+201 pp. $4.00.

Geometry for Advanced Pupils. By E. A. Maxwell. Oxford, England, Clarendon Press, 1949. 176 pp.

Plane Trigonometry—with Tables. By H. M. Dadourian. Cambridge, Addison-Wesley, 1950. 11+84 pp.

Elements of Analytic Geometry. By W. L. Hart. Boston, Heath, 1950. 9+264 pp. $2.75.

The Science of Chance. By H. C. Levinson. New York, Rinehart, 1950. 8+348 pp. $2.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITOR'S NOTE. During the period from January, 1948, to December, 1949, inclusive, this section of this MONTHLY listed 854 titles of papers presented to over one-hundred different clubs or societies. An attempt has been made to catalogue these titles but the variety of topics reported was so great that a summary appeared valueless.

Letters have been sent to the Chairman of Mathematics Departments urging that mathematics groups report their activities for 1949–50. Many clubs have responded and the reports will be published in the order received.

CLUB REPORTS, 1949–50

Kappa Mu Epsilon, Chicago Teachers College

The following papers were presented at meetings of the *Illinois Gamma* Chapter of *Kappa Mu Epsilon*:

Concepts of infinity and infinite sets, by Norman Stein.

Discriminants of polynomials, by Howard James.

In addition, a film on *The Einstein theory of relativity* was shown at a regular meeting.

Illinois Gamma Chapter was invited to sponsor a meeting in connection with the Twenty-Eighth Annual Convention of the National Council of Teachers of Mathematics. This meeting was held on April 14, 1950 at the Congress Hotel in Chicago. The National President, Prof. H. Van Engen, was present and opened the meeting with a brief speech. The program consisted of the following student papers:

A step forward, by Mary Lou Hodor and Nan Hutchings of the College of St. Francis, Joliet, Illinois.

Plane geometry without compasses, by Robert Allender of Iowa State Teachers College, Cedar Falls, Iowa.

Topology—modern supplement to Euclid, by Patricia Mitchell of Chicago Teachers College.

Some little known constructions with straight edge and compass, by Frances Donlon of Mount St. Scholastica College.

Inversion, by Dorothy Karner, of Mount Mary College, Milwaukee, Wisconsin.

The mathematical method in the light of a philosophical system, by Jeanne Culivan of Mount St. Scholastica College.

Providing for individual differences in arithmetic, by Dorothy Dahlberg of Chicago Teachers College.

The meeting was well attended and served the dual purposes of acquainting

members of Kappa Mu Epsilon with the National Council of Teachers of Mathematics and the reverse.

A party for entering freshmen, the annual banquet, and a spring picnic were the social events of the year.

The officers for 1949–50 were: President, Marie Marciante; Vice-President, William Woods; Secretary, Delphine Szulakiewicz; Treasurer, Manuel Sanchez; Sponsor, Dr. J. M. Sacks.

Pi Mu Epsilon, University of Oklahoma

The years activities of the *Oklahoma Alpha* Chapter of *Pi Mu Epsilon* included business, social, and regular meetings at which the following talks were given:

Lion chasing, by Donald Dubois

Mathematics and astonomy, by Prof. J. O. Hassler

The construction of the real number system, by Gene Levy

The application of mathematics to engineering problems, by Prof. J. Ray Matlock, School of Civil Engineering

Some aspects of mathematics related to naval sciences, by Forrest Grogan.

Ronald Smith was first place winner and T. L. McElhaney was second place winner of the annual, university-wide competitive mathematics examination. The chapter sponsored a tea for students and teachers of the state high schools that have entered students in the Interscholastic Algebra and Geometry Examinations.

An initiation banquet was held at which the guest speaker was Dean W. H Carson of the College of Engineering. A total of 26 new members were inducted into the chapter.

The following officers were elected: Director, Don A. Gorsline; Vice-Director, Walter C. Brown; Secretary-Treasurer, Rose M. Pratt.

Mathematics Club, McMaster University

The activities of the *Mathematics Club* of McMaster University for the academic year 1949–50 include the following topics, presented by members and guests:

Scale and notation, by Robert Attridge

A biography of Archimedes, by James McCulloch

Greek mathematics, by Lorraine McDiarmid

Laplace, by Edward Pennington

Mathematical instruments, by Richard Beesack

Experimental errors, by Joseph O'Brien

Life of Fourier, by Ross McFarlane

Fourier analysis with applications, by Vern Turner

Lorentz transformations and the Minkowski four-dimensional space-time continuum, by R. J. Donnelly

"*It is obvious*," by James Eastrott

Topology, By Dr. Krieger of the University of Toronto
Why does applied mathematics work?, by Dr. J. D. Bankier.

The officers for the coming year are: President, R. J. Donnelly; Vice-President, Robert Attridge; Secretary-Treasurer, U. C. Strahlendorf; Year Representatives, Robert Stevenson, James McCulloch, and Ross McFarlane.

Pi Mu Epsilon, Louisiana State University

The *Louisiana Alpha* Chapter of *Pi Mu Epsilon* held the following meetings during the year:

Newton's law of gravity, by Dr. F. A. Rickey
Statistical analysis of data, by Dr. Donald Billings
Length and area, by Dr. H. L. Smith
Transfinite numbers, by Dr. N. E. Rutt
Conservation of systems in phase space, Mr. Ernest Ikenberry
Mathematics as applied to chemistry, by Dr. Olen A. Nance.

Thirty-eight new members were initiated during the year. The Pi Mu Epsilon Lectures were given by Dr. H. S. Householder, Head of the Mathematics Section of Oak Ridge National Laboratory. His subject was, *The relation of mathematics to the sciences*. He also spoke on the subject *Statistics and the computer's art*.

The Senior Award was won by Miss Katherine Rollins and the Freshman Award was won by Mr. Van Be Luong. Five new volumes were added to the Pi Mu Epsilon shelf in the Mathematics Library.

The officers for the year were: President, Marion Smith; Vice-President Rodger Richardson; Secretary, Alice Pecot; Treasurer, Agnes Schertz; Faculty Advisor and Corresponding Secretary, Dr. H. T. Karnes.

Mu Alpha Theta, Yeshiva College

The Yeshiva College Mathematics Club, *Mu Alpha Theta*, devoted its program to lectures, delivered by the students, on various branches of geometry as follows:

Foundations of geometry (2 lectures), by Mr. Frank
Non-Euclidean geometry (3 lectures), by Mr. Gold
Analytic projective geometry (3 lectures), by Mr. Rosenfeld
Axiomatic projective geometry (2 lectures), by Mr. Solomon
Combinatorial topology (2 lectures), by Mr. Frank
Differential geometry (2 lectures), by Mr. Hellerstein.

Miss Ester Adler of Columbia University, spoke on *Godel's system of axioms sigma*.

Joint meetings were held with the Classical Society (on Pythagoras and Greek mathematics) and with the Hebrew Club (on mathematical terms in Hebrew).

Other activities included a number of formal and stag parties; a colloquium during the Passover recess; a picnic at the end of the school year; and the publication of a journal devoted to the original researches of the members.

The present officers of the club are: Presidents, Seymour Huber (1st semester) and Azriel Rosenfeld (2nd semester); Vice-President, Gustave Solomon; Secretary, Simon Hellerstein.

Kappa Mu Epsilon, Mount St. Scholastica College

The *Kansas Gamma* Chapter of *Kappa Mu Epsilon* at Mount St. Scholastica College had an enrollment of twenty-three members. The meetings were held bi-monthly. Besides the discussion meetings, the Chapter held the annual formal Christmas Wassail party, the pledge program, and the formal initiation and banquet.

In line with the general theme of the year, *Mathematics as a system of thought*, the following papers were presented:

Status of mathematics with the Greek philosophers, by Sister Helen Sullivan, O.S.B.

Mathematics in the patristic period, by Noreen Hurter

Mathematics in the scholastic period, by Mary Alice Weir

Contemporary mathematical philosophers, by Jeanne Culivan

Demonstration of mathematical systems, by Patti Shideler

Korean mathematics, by Peter Kim, guest speaker (Korea)

Geometric constructions with straight edge and compass, by Frances Donlon

Frequency distributions, by Frances Walsh

Various number systems, by Victoria Fritton

Our mathematical heritage, panel discussion by Margot Acree, Terry Breitenbach, and Dorothy Ripley.

The Chapter sent two delegates to Chicago to attend the convention of the National Council of Mathematics Teachers. The delegates read original papers in the Kappa Mu Epsilon section of the convention. Miss Frances Donlon read the paper, *Geometrical constructions with straight edge and compass*, while Miss Jeanne Culivan's paper was *The mathematical method in the light of a philosophical system*.

The tenth anniversary of the founding of *Kansas Gamma* on this campus was celebrated by a founder's day dinner. Testimonial letters and telegrams from charter members and past presidents were read.

The Hypatian Award was granted to Jeanne Culivan and Frances Donlon. Anne Robben won the underclassman Scholarship award.

The new officers of 1950–51 are: President, Frances Donlon; Vice-President, Jeanne Culivan; Secretary, Terry Breitenbach; Treasurer, Ruth Link; Sponsor, Sister Helen Sullivan, O.S.B.

Mathematics Club, College of the Holy Cross

The Holy Cross *Mathematics Club* conducted a series of seminars during the academic year 1949–50. The meetings were held twice monthly under the direction of Professors C. G. Schilling and V. O. McBrien. The following papers were

presented:

The gamma and beta functions, a series of four lectures, by Prof. V. O. McBrien

Theory of infinite products (two seminars)

The theorem of mean value

Unusual substitutions as a method of integration

Parametric method in maximum and minimum problems

Vertex and axis of a parabola in general position

The Heine-Borel covering theorem.

Solutions to proposed problems involving infinite products were presented by C. T. Mullins, J. A. Gobes, and W. F. Reynolds.

Pi Mu Epsilon, University of Delaware

The *Delaware Alpha* chapter of *Pi Mu Epsilon* heard the following talks during 1949–50:

Topological dynamics, by Russell Remage

Partitions, by Prof. N. J. Fine of the University of Pennsylvania

The brachistochrone problem, by A. C. Nelson.

Professor Fine spoke at the annual banquet in March.

Officers for 1950–51 are: President, Patricia Reybold; Secretary, Donald Clark; Treasurer, E. K. Rees; Social Chairman, Mervine Rosen; Program Chairman, Martha Walker; Director, Russell Remage.

Kappa Mu Epsilon, Westminster College

An important event in the past year for *Delta Nabla*, Westminster honorary mathematics society, was their installation as the *Pennsylvania Alpha* chapter of *Kappa Mu Epsilon*. At this ceremony, conducted by Dr. L. F. Ollmann, National Treasurer, nineteen students and four faculty members were initiated into *Kappa Mu Epsilon*.

The outstanding papers presented to the club were:

Linkages, by Fred Cassell and Robert Gunnett

I. B. M. computation, by Mrs. Ben Bolt

Abacus, by Thomas Petrecca.

During the year, the club donated to the book drive for the library and also donated a book to the outstanding mathematics student now in the Sophomore class. The award was made on the basis of mathematics grades and general average during the Freshman year.

Banquets held at Christmas and in the Spring, served as opportunities for the initiation of new members. At these ceremonies, each new member took the name of a famous mathematician and gave a short history of that person's life.

The present officers of the Pennsylvania Alpha chapter of Kappa Mu Epsilon are: President, John Williamson; Secretary, Barbara Williams; Vice-President, John Kratz; Treasurer, Henry Barnhart; Faculty Advisor, Prof. G. W. Conway, Jr.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ANNUAL MEETING OF THE A.A.A.S.

The Annual Meeting of the American Association for the Advancement of Science will be held on December 26–30, 1950, in Cleveland, Ohio. All of the sections and subsections of the A.A.A.S. and more than forty participating organizations have completed plans for more than two hundred sessions. An extensive series of tours to museums, laboratories, and industrial plants of the Cleveland area has been planned. The Annual Science Exposition will be held in the Arena of Cleveland's Public Auditorium.

The Cleveland Convention Bureau is in charge of housing arrangements. Announcements and coupons for reservations will be found in recent issues of *Science* and *The Scientific Monthly*. Advance registration closes on December 4, 1950.

STANFORD UNIVERSITY COMPETITIVE EXAMINATION IN MATHEMATICS

The fifth Stanford University Competitive Examination in Mathematics (see this MONTHLY, vol. 53, no. 7, pp. 406–409 (1946)) was held March 25, 1950 in 34 high schools in California; 207 senior high school students participated. The following problems were proposed:

1. Observe that

$$1 = 1$$
$$1 - 4 = -(1 + 2)$$
$$1 - 4 + 9 = 1 + 2 + 3$$
$$1 - 4 + 9 - 16 = -(1 + 2 + 3 + 4)$$

Guess the general law suggested by these examples, express it in suitable mathematical notation, and prove it.

2. Given a square. Find the locus of the points from which the square is seen under an angle (a) of 90° (b) of 45°. (Let P be a point outside the square, but in the same plane. The smallest angle with vertex P containing the square is the "angle under which the square is seen" from P.) Sketch clearly both loci, give a full description, and a proof.

3. Call "axis" of a solid a straight line joining two points of the surface of the solid and such that the solid, rotated about this line through an angle which is greater than 0° and less than 360° coincides with itself.

A cube has 13 different axes, which are of three different kinds. Describe clearly the location of these axes, find the angle of rotation associated with each. Assuming that the edge of the cube is of unit length, compute the arithmetic mean of lengths of the 13 axes. Do not use tables and compute to two decimals.

The writer of the best paper, Glen Bredon, student at Sanger Union High School, Sanger, California, received a $600 Scholarship at Stanford University. Also, John R. Berry of Berkeley, California, Richard Couchman of Santa Ana, California, and John Rosser of Oakland, California, received "Honorable Mention."

PERSONAL ITEMS

Professor R. E. Langer of the University of Wisconsin and Professor L. M. Graves, University of Chicago, were the delegates of the Association to the International Congress of Mathematicians which was held at Cambridge, Massachusetts, on August 30–September 6, 1950.

Professor Emeritus F. W. Owens of Pennsylvania State College was the representative of the Association at the inauguration of President M. S. Eisenhower of Pennsylvania State College on October 5, 1950.

Associate Professor P. C. Rosenbloom of Syracuse University has been awarded the Fréchet Prize by the French Academy of Sciences for a paper pertaining to applications in abstract spaces.

Professor John von Neumann of the Institute for Advanced Study has been awarded honorary degrees of Doctor of Science by Harvard University and the University of Pennsylvania.

Case Institute of Technology announces: Dr. R. A. Clark, Massachusetts Institute of Technology, has been appointed to an instructorship; Mr. Philip Clyne of Mid-Essex Technical College, Chelmsford, England, has been appointed Visiting Instructor for the academic year 1950–51; Dr. R. F. Rinehart, who has been serving as Executive Secretary of the Research and Development Board, Washington, D. C., has resumed the position of Professor of Mathematics.

Central Michigan College reports the following: Mr. August Arndt, formerly principal of Fenton High School, Michigan, has been appointed Visiting Assistant Professor; Mr. Dana Sudborough has been granted a year's leave of absence for the purpose of doing graduate work at Michigan State College.

Kansas State College makes the following announcements: Dr. Abraham Franck of the University of Minnesota has been appointed to an assistant professorship; Dr. Violet G. Hachmeister, formerly an instructor in the Extension Division of the University of Wisconsin, has been appointed to an assistant professorship.

At Randolph-Macon Woman's College: Associate Professor M. Gweneth Humphreys has been promoted to the position of Professor and Head of the Department of Mathematics; Professor Gillie A. Larew, formerly head of the Department of Mathematics, is continuing as Professor of Mathematics and Dean of the College.

Stanford University reports that the following persons delivered lectures in the Mathematics Department during the Summer Quarter, 1950: Dr. Paul Erdös, Aberdeen, Scotland; Professor Theodor Estermann, University College, London; Professor M. Fekete, University of Jerusalem; Professor K. A. Hirsch,

Durham University, King's College, Newcastle, England; Professor Paul Levy, Ecole Polytechnique, Paris; Professor Walter Saxer, Polytechnical School, Zurich, Switzerland.

Syracuse University announces the following: Dr. W. A. Pierce, formerly graduate student at Harvard University, has been appointed to an assistant professorship; Assistant Professor P. C. Rosenbloom has been promoted to an associate professorship; Mrs. May N. Harwood has retired with the title of Professor Emeritus.

At the University of Delaware: Dr. Russell Remage, Jr. has been promoted to an assistant professorship; Dr. Henry Teicher has been appointed to an assistant professorship.

University of Minnesota reports the following: Dr. M. D. Donsker of Cornell University has been appointed to an assistant professorship; Assistant Professor Elizabeth S. Carlson has been promoted to an associate professorship; Miss Ella Thorp has been promoted to an assistant professorship.

University of Nebraska announces the following appointments to assistant professorships: Dr. L. K. Jackson of the University of California at Los Angeles and Dr. George Seifert of Cornell University.

University of South Carolina makes the following announcements: Associate Professor E. A. Hedberg has been promoted to a professorship; Instructors C. F. Martin, Lillian Perkins, and Betty Weber have been promoted to assistant professorships; Associate Professor J. D. Novak is on leave and is doing graduate work at Michigan State College; Assistant Professor R. R. Croxton is on leave of absence and is located at the University of North Carolina; Instructor C. W. Huff is also on leave of absence and is engaged in graduate study at Tulane University.

University of Toledo reports: Professor H. L. Zeiders who has been in military service in Germany has returned to his teaching position; Professor O. L. Dustheimer and Miss Lois L. Martin have resigned.

University of Tulsa announces the following appointments to instructorships: Dr. R. N. Goss, formerly instructor at Iowa State College, and Mr. H. G. Shaklee, previously assistant at Oklahoma University.

Assistant Professor E. H. Allen of Springfield College has been appointed Registrar of Northeastern University, Springfield Extension.

Dr. R. D. Anderson, University of Pennsylvania, has been promoted to an assistant professorship.

Assistant Professor T. W. Anderson of Columbia University has been promoted to an associate professorship.

Associate Professor H. M. Bacon of Stanford University has been promoted to a professorship.

Mr. Max Beberman, formerly a graduate assistant at Teachers College, Columbia University, has been appointed to an instructorship at University High School, University of Illinois.

Associate Professor J. H. Bell of Michigan State College is serving as Acting

Head of the Department of Mathematics during the year 1950–51.

Mr. C. R. Blyth, research assistant at the University of California at Berkeley, has been appointed to an assistant professorship at the University of Illinois.

Professor Harald Bohr of the University of Copenhagen, Denmark, delivered a series of lectures at the University of Georgia on September 13–18, 1950.

Miss Marguerite A. Brown, recent graduate of Carleton College, has been appointed Teaching Assistant at the University of Minnesota.

Mr. Errol Buker, a teacher in the Springfield Public Schools, has received an appointment to an instructorship at Springfield College.

Dr. F. E. Clark of Tulane University has been appointed to an assistant professorship at University College, Rutgers University.

Instructor H. C. Cooke of North Carolina State College has been promoted to an assistant professorship.

Mr. R. B. Derflinger is teaching in the Department of Chemistry of Turtle Creek High School, Turtle Creek, Pennsylvania.

Mr. G. M. Dillon, statistician in the Treasury Department of E. I. DuPont de Nemours Company, Wilmington, Delaware, is now in military service and is located at Camp Gordon, Augusta, Georgia.

Dr. W. B. Evans of the University of Illinois has been appointed to an assistant professorship at the Georgia Institute of Technology.

Associate Professor G. M. Ewing of the University of Missouri has been promoted to a professorship.

Dr. H. M. Feldman, formerly a lecturer at Washington University, is now teaching at Beaumont High School, University City, Missouri.

Miss Ruth A. Fish of the University of Arizona is teaching at Contra Costa Junior College, Richmond, California.

Assistant Professor Evelyn Frank of the University of Illinois has been promoted to an associate professorship.

Dr. J. R. Garrett of Duke University has been appointed to an assistant professorship at Georgia Institute of Technology.

Assistant Professor R. D. Gordon of the University of Buffalo has been appointed Professor and Head of the Department of Mathematics of Washington College.

Assistant Professor J. B. Greeley has been promoted to an associate professorship and is continuing to serve as Head of the Department of Mathematics of Utica College, Syracuse University.

Assistant Professor R. E. Greenwood of the University of Texas has been recalled to active duty by the Navy Department.

Mr. W. H. Gregory is employed as a teacher at Hayward Union High School, Hayward, California.

Mr. F. D. Grogan, previously teaching assistant at the University of Oklahoma, is teaching now at Minden High School, Minden, Louisiana.

Dr. Emil Grosswald of the University of Pennsylvania has been appointed

Lecturer at the University of Saskatchewan.

Dr. A. J. Hoffman of Columbia University is now a member of the Institute for Advanced Study.

Dr. R. V. Hogg of the State University of Iowa has been promoted to an assistant professorship.

Instructor J. R. Holzinger of Franklin and Marshall College has been promoted to an assistant professorship.

Miss Marian B. Jeffries, graduate student of the University of Virginia, has been appointed to an instructorship at the University of Richmond.

Associate Professor N. D. Lane of Acadia University has been appointed Lecturer at Carleton College, Ottawa, Canada.

Mr. H. R. Leifer of the Registration and Research Section, Veterans Administration, Pittsburgh, has been promoted to the position of Chief of the Section.

Assistant Professor R. C. Luippold of Antioch College has been appointed to an instructorship at New York State College for Teachers, Albany.

Dr. Spencer Macy has been appointed Assistant Professor of Mathematical Physics at the South Dakota School of Mines and Technology.

Associate Professor W. C. McDaniel of Southern Illinois University has been promoted to the position of Professor and Chairman of the Department of Mathematics.

Mr. J. W. Mettler of Lehigh University has accepted a position with the Educational Testing Service, Princeton, New Jersey.

Mr. P. T. Mielke, industrial research fellow at Purdue University, has been appointed to an assistant professorship at Wabash College.

Associate Professor E. P. Northrop, who is also Associate Dean of the College of the University of Chicago, has been promoted to a professorship.

Dr. Margaret Owchar of the University of Minnesota has been appointed to an associate professorship at Southwest Missouri State College.

Dr. Jesse Pierce has been appointed Research Engineer at Northrop Aircraft, Hawthorne, California.

Assistant Professor A. L. Putnam has been promoted to an associate professorship in the College of the University of Chicago.

Mr. V. J. Rathbun, formerly head of Science Department of Colegio Ponceno de Varones, Ponce, Puerto Rico, is now a graduate student at Catholic University.

Mr. T. L. Reynolds, previously a part-time instructor at the University of North Carolina, has been appointed to the position of Professor and Head of the Department of Mathematics of Millsaps College.

Associate Professor H. E. Robbins of the University of North Carolina has been promoted to a professorship.

Mr. F. E. Ross of the University of North Carolina has accepted a position as auditor in the Division of Rates and Services, Department of Transportation, Frankfort, Kentucky.

Miss Augusta L. Schurrer, graduate assistant at the University of Wisconsin, has been appointed to an instructorship at Iowa State Teachers College.

Mr. W. H. Sellers, formerly teaching fellow at West Virginia University, has been appointed Head of the Department of Mathematics of Alderson-Broaddus College.

Sister Mary Ferrer, previously a graduate student at Notre Dame University, is now Head of the Department of Mathematics at St. Francis Xavier College for Women.

Associate Professor R. W. Wagner of Oberlin College has been appointed to a professorship at the University of Massachusetts.

Mr. A. M. Weitzenhoffer has accepted a position as research associate at the College of Dentistry, University of Detroit.

Associate Professor William Beverley of Lafayette College died on July 24, 1950.

Professor H. L. Smith of Louisiana State University died on June 13, 1950. He was a charter member of the Association.

Mr. T. E. Tuthill, head of the Department of Mathematics of Nichols School, Buffalo, New York, died on August 22, 1950.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

SUMMER MEETING OF THE BOARD OF GOVERNORS

Although the usual Summer Meeting of the Mathematical Association of America was omitted because of the International Congress of Mathematicians, the Board of Governors of the Association met on the evening of Thursday, August 31, 1950, in Phillips Brooks House of Harvard University, Cambridge, Massachusetts. Twenty-six of the forty-one members of the Board were present. Among the more important items of business transacted were the following:

Since the Committee on an Index of the MONTHLY has completed its task, the Committee was discharged with a vote of thanks for its labors on behalf of the Association. It was also voted to express the thanks of the Association to Professors C. B. Allendoerfer, C. O. Oakley, D. L. Thomsen, and A. H. Wilson of Haverford College, and C. P. Bruderle of Villanova College.

It was voted to accept the invitation of the Mathematics Division of the American Society for Engineering Education to hold a joint meeting at Michigan State College, East Lansing, Michigan, in June 1951.

Reports of progress were submitted by the Committee on a Guidance Pamphlet, the Committee on Slaught Memorial Papers, and the Director of the Putnam Prize Competition. It was voted to discharge the standing Committee for the Coordination of Studies in Mathematical Education and to change the name of the standing Committee on Section Meetings to the Committee on Sections. Reports were also received from the Committee on Cooperation with

Industry and from the representatives on the Policy Committee for Mathematics.

H. M. GEHMAN, *Secretary-Treasurer*

MATHEMATICS CONTEST OF THE METROPOLITAN NEW YORK SECTION

On May 11, 1950, the Metropolitan New York Section of the Associat on conducted a mathematical contest open to high school students in the area of the Section. A total of 690 students from 238 high schools participated in the competition.

The Bronx High School of Science, whose students scored 437 out of a possible 450 points, was awarded the Association's bronze cup. Honorable mentions were won by Brooklyn Technical High School, Abraham Lincoln High School and Marymount School.

First place individual honors were awarded to Paul H. Monsky of Brooklyn Technical High School for presenting a perfect paper. Individual honorable mentions go to Arnold M. Faden, Lee R. Abramson, and Samuel J. Klein, all of the Bronx High School of Science.

Certificates of Merit for proficiency in Secondary School Mathematics were awarded to the highest ranking student in each of the eleven regions to which the participating schools were assigned. Also the top student in each school received a gold medal in the form of a lapel button or pin bearing a facsimile of the seal of the Mathematical Association of America.

The management of the contest was undertaken by the Section's Committee on Contests and Awards consisting of Brother Bernard Alfred, W. H. Fagerstrom (Chairman), H. F. Fehr, Julius Freilich, J. J. Kinsella, V. S. Mallory, K. B. Morgan.

The contest included material only from Elementary Algebra, Plane Geometry and Intermediate Algebra. Copies of the contest questions and other material used in publicizing the contest may be obtained from Professor W. H. Fagerstrom, City College of the City of New York, New York 31, New York.

The engraved Certificate of Merit and the gold medals are available to any Section wishing to conduct a contest of this type. For information consult the Secretary-Treasurer of the Association.

ANNUAL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Kentucky, on Saturday, April 29, 1950. Dr. H. H. Downing, Chairman of the Section, presided at the morning and afternoon sessions.

Sixty-four persons were present at the meeting, including the following thirty-two members of the Association: D. F. Atkins, P. P. Boyd, M. C. Brown, Esther A. Compton, A. E. Cook, Lake C. Cooper, V. F. Cowling, D. L. Daly, H. H. Downing, Clarence Ford, A. E. Foster, A. W. Goodman, Beulah Graham, Charles Hatfield, Jr., A. S. Householder, Aughtum S. Howard, Tadeusz

Leser, W. L. Moore, E. J. Musch, R. S. Park, D. W. Pugsley, Sara L. Ripy, G. G. Roberts, W. J. Robinson, Florence V. Rohde, W. C. Royster, Klaus Shocken, D. E. South, Guy Stevenson, J. T. Vallandingham, J. A. Ward, Jean M. Wyre.

Officers elected at the meeting were: Chairman, R. S. Park, Eastern Kentucky State College; Secretary, Aughtum S. Howard, Kentucky Wesleyan College. The annual meeting in 1951 is to be held on Saturday, April 28, at Eastern Kentucky State College, Richmond, Kentucky.

The following papers were presented:

1. *On the coefficient problem for univalent functions*, by Professor A. W. Goodman, University of Kentucky.

A function is called univalent in the open unit circle if it is regular there and assumes no value more than once. A brief survey is given of the present state of the problem of determining the upper bound for the absolute value of the coefficients. The corresponding problem for p-valent functions is also discussed.

2. *To find which and how many of the numbers formed from permutations of the digits 1–9 are divisible by eleven*, by Professor Paul C. Overstreet, Morehead State College, introduced by the Secretary.

Professor Overstreet developed a theory which enabled him to identify those integers formed by permutations of the numbers 1, 2. 3, 4, 5, 6, 7, 8, 9 which are divisible by 11. He found that there are 28,800 such integers.

3. *Basic and atomic propositions in mathematics and physics*, by Dr. Klaus Schocken, Medical Department, Field Research Laboratory, Fort Knox, Kentucky.

Basic propositions and atomic propositions are both essential in science. Their functions, however, are entirely different. The question then arises: What are the roles which basic propositions, atomic propositions, and logical propositions play in mathematics and physics? It is obvious that mathematics contains no basic propositions, because it is tautological. Since it can be derived entirely from logic, mathematics consists of logical formulae.

On the other hand, basic propositions are the foundation of physics. The records, which the experimenter takes, consist of basic propositions. These basic propositions appear rarely, almost never, in a completed physical theory. The propositions of a physical theory are always logical formulae, which have been derived from the basic propositions by generalization.

A completed physical theory consists of logical formulae (axioms) which serve as premises for the conclusions. The premises contain not only variables but also individual predicates and individual objects. It is then possible to derive from the axioms the atomic formulae, of which they are compounded. A section of physics, which is accessible to axiomatic treatment, needs the axioms themselves and also the logical theorems, which make possible the derivation of the conclusions from the axioms. In almost all cases the theorems of mathematics will be sufficient for this purpose; in some cases theorems of theoretical logic will have to be added to them. The atomic formulae, at which we thus arrive, find their justification in basic propositions which can be derived from the atomic propositions. This derivation of experimental results from theoretical constructions has not the same rigor as the derivation of atomic formulae from logical formulae. A basic proposition may be derived from a great many different premises, and therefore a basic proposition can never verify an atomic formula. It can only make its validity plausible.

4. *Notes from national meetings*, by Professor M. C. Brown, University of Kentucky.

Topics dealing with section meetings, special activities of the sections, and membership were presented.

5. *The golden section*, by Professor Aughtum S. Howard, Kentucky Wesleyan College.

The development of the golden section within the Pythagorean school, and its influence on the arts and culture of the golden age of Greece, were outlined. The relation between the golden section and the Fibonacci series as developed in the middle ages and applied to the study of the inflorescence of pine cones, etc., was discussed.

6. *Slide rules*, by Professor W. A. Ballou, Morehead State College, introduced by the Secretary.

Professor Ballou discussed the history of the development of the theory of logarithms, the compilation of log tables, and the construction of slide rules. He displayed models of slide rules manufactured in different countries and varying greatly in design.

7. *On some recent developments in continued fractions*, by Professor V. F. Cowling, University of Kentucky.

In this paper the recent developments in convergence criteria for continued fractions were discussed. Emphasis was placed on the geometric theory as developed by Leighton and Thron, and in particular on the value region problem.

8. *Division by zero is possible*, by Professor W. L. Moore, University of Louisville.

Using the definitions and theorems as given in Veblen and Young, Vol. 1, Chapter VI, Dr. Moore pointed out that the division of P_m, $m \neq 0$, by P_0 was unique, and gave the point P_∞ on the scale of points. It is only when m itself is zero that an indeterminancy results. He then pointed out that in the classical proof that division by zero is impossible in algebra that, if in the next to the last step, we put $m/0 = \infty$ for $m \neq 0$ then this step will be

$$\frac{a+b}{a-b}(a-b) = \frac{b}{a-b}(a-b), \quad \text{or} \quad \infty\,(a-b) = \infty\,(a-b),$$

and no contradiction results giving the usual $2 = 1$. The contradiction as usually found results in dividing $(a-b)/(a-b)$ which, even in the algebra of points, is an indeterminant quantity.

9. *Euclid's algorithm and the least-remainder algorithm*, by Professor A. W. Goodman and Mr. W. M. Zaring, University of Kentucky.

These speakers considered the problem of the number of equations necessary to find the greatest common divisor of two integers by the least-remainder algorithm as compared with the Euclid algorithm. It was proved that the number of equations required by the least-remainder algorithm is less than the number required by the Euclid algorithm by exactly the number of negative signs in the least-remainder algorithm.

10. *The inversion of large matrices*, by Dr. A. S. Householder, Oak Ridge National Laboratory, Oak Ridge, Tennessee.

On the basis of simple geometrical considerations a class of iterative methods can be deduced for inverting matrices and for solving systems of linear algebraic equations. Special cases include the methods of steepest descent, of relaxation, and of Gauss and Seidel, as applied to positive definite matrices, and certain natural generalization of these for general matrices.

AUGHTUM S. HOWARD, *Secretary*

SPRING MEETING OF THE SOUTHWESTERN SECTION

The tenth annual meeting of the Southwestern Section of the Mathematical Association of America was held at Arizona State College, Flagstaff, Arizona, on April 29, 1950.

Professor E. J. Purcell, Chairman of the Section, presided at the morning and afternoon sessions. Thirty-six persons attended the sessions, including the following twenty-three members: J. W. Beach, A. W. Boldyreff, C. E. Buell, J. H. Butchart, Lee Byrne, W. W. Denton, F. C. Gentry, R. F. Graesser, M. S. Hendrickson, R. C. Hildner, Max Kramer, Lincoln La Paz, Rachael La Roe, B. C. Meyer, E. J. Purcell, B. D. Roberts, H. P. Rogers, J. L. Slechticky, R. D. Stalley, Deonisie Trifan, Earl Walden, D. L. Webb, and Charles Wexler.

At the business session the following officers were elected for the year 1950–51: Chairman, A. W. Boldyreff, University of New Mexico; Vice-Chairman, Charles Wexler, Arizona State College, Tempe, Arizona; Secretary-Treasurer, R. L. Westhafer, New Mexico College of A. and M. A.; Lecturer, one-year term, J. H. Butchart, Arizona State College, Flagstaff, Arizona; two-year term, M. S. Hendrickson, University of New Mexico. The section accepted the invitation of the University of New Mexico to hold the spring 1951 meeting in Albuquerque. An assessment of $1.00 per member was voted.

At an evening banquet session Dr. Earl C. Slipher of Lowell Observatory delivered an illustrated lecture on the planet Mars.

Fifteen papers were crowded into the one-day sessions:

1. *An extension of Eisenstein's criteria of irreducibility*, by D. L. Webb, of the University of Arizona.

The speaker presented an extension of Eisenstein's criteria of irreducibility of polynomials with regard to linear factors, treating the case when p^m is a factor of the constant term.

2. *The place of statistics in the curriculum*, by J. W. Beach and H. P. Rogers, University of New Mexico.

3. *Short formulations of Boolean algebra, using ring operations*, by Dr. Lee Byrne, Arizona State College, Tempe.

Much interest has attached to recent formulations of Boolean algebras intended to emphasize their character as rings, and thus featuring especially ring operations. Most of these are relatively long, and Dr. Byrne's note was addressed to the question whether formulations of this type could not be made as brief as those representing other modes of approach. Leaving closure (and non-emptiness) assumptions tacit, and using as undefined concepts ring addition and multiplication and a special element 1 (one), he presented four "transformation" axioms: I. $(X+Y)+Z=X+(Y+Z)$; II. $(X+Y)+X=Y$; III. $X(W+YZ)=XW+Y(ZX)$; IV. $X(X+1)=Y+Y1$. From these are derived nine theorems sufficing to show the system to be a Boolean group, a ring, a Boolean ring, and finally a Boolean algebra. He further set up an alternative set of two such axioms, II as already given and the new I′: $X+X=P_1+(P_2+P_3)$ (employing a device due to B. A. Bernstein), where P_1, P_2, and P_3 are respectively the abbreviations $P_1=[(A+B)+C]+[A+(B+C)]$, $P_2=C(F+DE)+[CF+D(EC)]$, $P_3=G(G+1(+(H+H1)$. After deducing three of the old theorems and two new ones it is then possible to deduce the previous four axioms from the two last mentioned.

4. *An inverse problem of the calculus of variations for multiple integrals*, Lincoln La Paz, University of New Mexico.

5. *On proofs of Stokes' theorem, involving Taylor's expansion*, W. W. Denton, University of Arizona.

In most text books on mathematical physics, the proofs of Stokes' formula, Gauss' formula, the relation of curl to angular velocity, the equation of continuity, and similar theorems are, in general, rather poorly presented. The proofs in question involve the expansion of functions by Taylor's formula, and integration of results over an infinitesimal curve or surface. The object of this paper is to explain why these particular proofs are weak, and to point out how they may be strengthened. The improvements suggested depend upon the following device. With each point of the given vector field there is associated an auxiliary system of coordinates having that point as origin, and the integration over the infinitesimal curve or surface is actually carried out.

6. *The boundaries of an arbitrary function*, M. S. Hendrickson, University of New Mexico.

7. *Invariant spherical harmonics*, B. C. Meyer, University of Arizona.

Let G be a finite group of orthogonal linear transformations of the variables X, Y, Z. A generating function was developed which gives the number of linearly independent spherical harmonics of a given degree M which are invariants of G.

8. *An arithmetical triangle*, R. D. Stalley, University of Arizona.

A triangular array of numbers is presented as it arises in the summation of a generalized geometric series. Properties and further applications of the triangular array are discussed.

9. *Some diophantine equations*, A. W. Boldyreff, University of New Mexico.

10. *A range of conics associated with a system of coaxial circles*, J. H. Butchart, Arizona State College, Flagstaff.

Professor Butchart considered the figure formed by drawing tangents from a variable point of a circle of a non-intersecting system of coaxial circles to two arbitrary circles of the same system so that both of the chosen circles are within the angle formed by the tangents. When the points of contact are joined, the circles determine chords on this line which have a constant ratio. The envelope of this line is an ellipse whose major auxiliary circle is another circle of the system. As other circles of the system are considered as loci of the variable point, other ellipses are formed all of which are tangent to the four common tangents to the two chosen circles.

11. *A non-involutorial Cremona transformation, in four-dimensional space*, E. J. Purcell, University of Arizona.

12. *Mathematics in meteorology*, Max Kramer, New Mexico College of A. and M. A.

Requirements for mathematical forecasting were discussed, and the curriculum pursued by the average meteorologist outlined. Seven tables compiled from a thorough analysis of the mathematical content of textbooks and other publications covering the entire meteorological field were presented with comment. Three important problems were solved to exhibit the type of mathematics required. In each case the theorems were derived, using both cartesian and vector techniques, and results applied to the following meteorological problems: (1) The Eulerian form of the equations of motion; (2) Bjerknes's circulation theorem; (3) The variation of wind in the friction layer, leading to the logarithmic spiral due to Ekman.

13. *An infinite set of formulas connecting binomial coefficients*, R. F. Graesser, University of Arizona.

14. *The dilemma in mathematics*, Charles Wexler, Arizona State College, Tempe.

15. *The plastic bending of plates*, Deonisie Trifan, University of Arizona.

In this paper Dr. Trifan formulated the basic partial differential equation governing the mechanical behavior of thin plates subjected to distributed transverse loads which produce plastic deformations. The plates were considered homogeneous and composed of isotropic materials exhibiting a gradual transition from the elastic to the plastic state. The stress-strain relation used was

$$\dot{s}_{ij} = 2G\dot{e}_{ij} - (G_0 - G)\dot{E}/Ee_{ij}.$$

where s_{ij} and e_{ij} are tensors of stress and strain deviation, the elastic shear modulus and the tangent shear modulus are G_0 and G respectively, the latter being assumed as a function of the strain intensity $E = 2/3\ e_{ij}\ e_{ij}$, and variables with dots indicate rates of change with respect to time.

B. D. ROBERTS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

Joint Meeting with American Society for Engineering Education, Michigan State College, East Lansing, June 25–26, 1951.

Thirty-second Summer Meeting, University of Minnesota, Minneapolis, September 3–4, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May, 1951.

ILLINOIS, University of Illinois, Urbana, May 11–12, 1951.

INDIANA, May 5, 1951.

IOWA, Wartburg College, Waverly, April 20–21, 1951.

KANSAS

KENTUCKY, Eastern Kentucky State College, Richmond, April 28, 1951.

LOUISIANA-MISSISSIPPI, Mississippi State College, State College, February 16–17, 1951.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, December 9, 1950.

METROPOLITAN NEW YORK, Spring, 1951.

MICHIGAN, Michigan State College, East Lansing, March 24, 1951.

MINNESOTA, College of St. Benedict, St. Joseph, April 28, 1951.

MISSOURI, Central College, Fayette, Spring, 1951.

NEBRASKA, University of Nebraska, Lincoln, May 5, 1951.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, Columbus, April 21, 1951.

OKLAHOMA, Oklahoma City, November 13, 1950.

PACIFIC NORTHWEST, State College of Washington, Pullman, June 15, 1951.

PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.

ROCKY MOUNTAIN, Colorado State College of Education, Greeley, April, 1951.

SOUTHEASTERN, Vanderbilt University and Peabody College, Nashville, Tennessee, March 16–17, 1951.

SOUTHERN CALIFORNIA, Whittier College, Whittier, March 10, 1951.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1951.

TEXAS, Southern Methodist University, Dallas, Spring, 1951.

UPPER NEW YORK STATE, Hamilton College, Clinton, May 5, 1951.

WISCONSIN, Carroll College, Waukesha, May, 1951.

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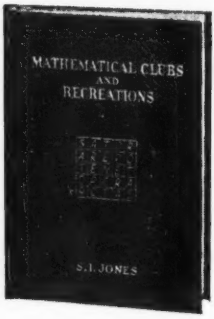

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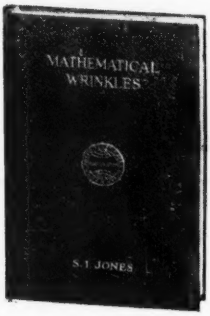

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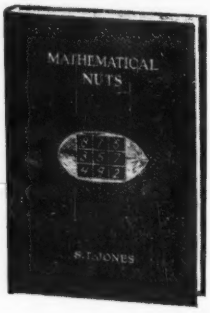

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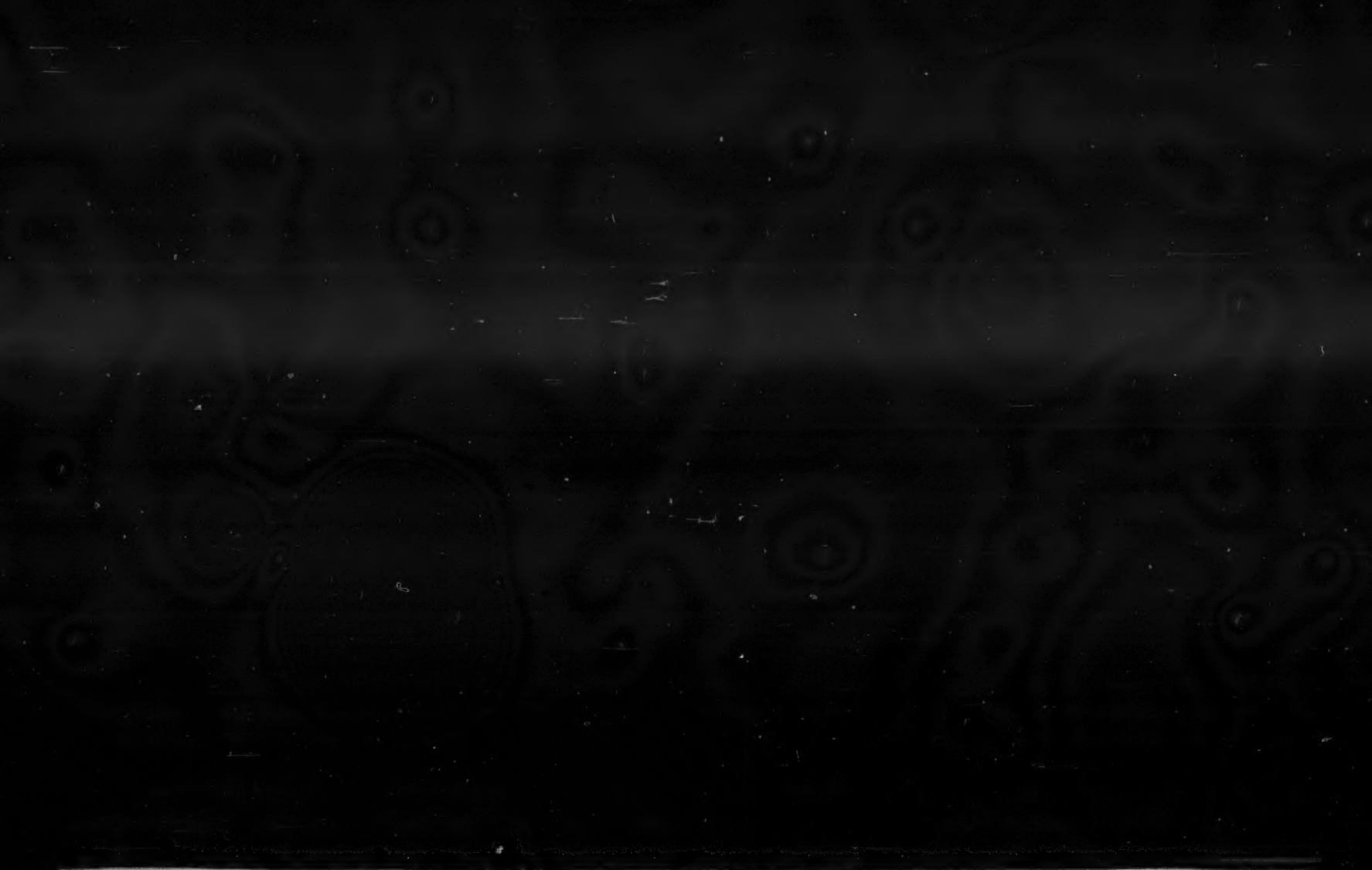

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